

# Beyond a question of Markus Linckelmann

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**Abstract:** In the 2002 Durham Symposium, Markus Linckelmann [1] conjectured the existence of a *regular central  $k^*$ -extension* of the full subcategory over the *selfcentralizing Brauer pairs* of the *Frobenius  $P$ -category*  $\mathcal{F}_{(b,G)}$  associated with a block  $b$  of defect group  $P$  of a finite group  $G$ , which would include, as  $k^*$ -automorphism groups of the objects, the  $k^*$ -groups associated with the *automizers* of the corresponding *selfcentralizing Brauer pairs*, introduced in [3, 6.6]; as a matter of fact, in this question the *selfcentralizing Brauer pairs* can be replaced by the *nilcentralized Brauer pairs*, still getting a positive answer. But the condition on the  $k^*$ -automorphism groups of the objects is *not* precise enough to guarantee the *uniqueness* of a solution, as showed in [2, Theorem 1.3]. This *uniqueness* depends on the *folder structure* [5, Section 2] associated with  $\mathcal{F}_{(b,G)}$  in [4, Theorem 11.32], and here we prove the *existence* and the *uniqueness* of such *regular central  $k^*$ -extension* for any *folded Frobenius  $P$ -category*.

## 1. Introduction

1.1. Let  $p$  be a prime number and  $\mathcal{O}$  a complete discrete valuation ring with a *field of quotients*  $\mathcal{K}$  of characteristic zero and a *residue field*  $k$  of characteristic  $p$ ; we assume that  $k$  is algebraically closed. Let  $G$  be a finite group,  $b$  a *block* of  $G$  — namely a primitive idempotent in the center  $Z(\mathcal{O}G)$  of the group  $\mathcal{O}$ -algebra  $\mathcal{O}G$  — and  $(P, e)$  a maximal *Brauer  $(b, G)$ -pair* [4, 1.16]; recall that the *Frobenius  $P$ -category*  $\mathcal{F}_{(b,G)}$  associated with  $b$  is the subcategory of the category of finite groups where the objects are all the subgroups of  $P$  and, for any pair of subgroups  $Q$  and  $R$  of  $P$ , the morphisms  $\varphi$  from  $R$  to  $Q$  are the group homomorphisms  $\varphi: R \rightarrow Q$  induced by the conjugation of some element  $x \in G$  fulfilling

$$(R, g) \subset (Q, f)^x \quad 1.1.1$$

where  $(Q, f)$  and  $(R, g)$  are the corresponding Brauer  $(b, G)$ -pairs contained in  $(P, e)$  [4, Ch. 3].

1.2. Moreover, we say that a Brauer  $(b, G)$ -pair  $(Q, f)$  is *nilcentralized* if  $f$  is a *nilpotent block* of  $C_G(Q)$  [4, 7.4], and that  $(Q, f)$  is *selfcentralizing* if the image  $\bar{f}$  of  $f$  is a block of *defect zero* of  $\bar{C}_G(Q) = C_G(Q)/Z(Q)$  [4, 7.4]; thus, a selfcentralizing Brauer  $(b, G)$ -pair is still nilcentralized. We respectively denote by  $\mathcal{F}_{(b,G)}^{\text{nc}}$  or by  $\mathcal{F}_{(b,G)}^{\text{sc}}$  the *full* subcategories of  $\mathcal{F}_{(b,G)}$  over the set of subgroups  $Q$  of  $P$  such that the Brauer  $(b, G)$ -pair  $(Q, f)$  contained in  $(P, e)$  is respectively nilcentralized or selfcentralizing.

1.3. Recall that a  $k^*$ -group  $\hat{G}$  is a group endowed with an injective group homomorphism  $\theta: k^* \rightarrow Z(\hat{G})$  [3, §5], that  $G = \hat{G}/\theta(k^*)$  is called the

$k^*$ -quotient of  $\hat{G}$  and that a  $k^*$ -group homomorphism is a group homomorphism which preserves the “multiplication” by  $k^*$ ; let us denote by  $k^*\text{-Gr}$  the category of  $k^*$ -groups with finite  $k^*$ -quotient. In the case of the Frobenius  $P$ -category  $\mathcal{F}_{(b,G)}$  above, for any nilcentralized Brauer  $(b, G)$ -pair  $(Q, f)$  contained in  $(P, e)$  it is well-known that the action of  $N_G(Q, f)$  on the simple algebra  $\mathcal{OC}_G(Q)f/J(\mathcal{OC}_G(Q)f)$  supplies a  $k^*$ -group  $\hat{N}_G(Q, f)/C_G(Q)$  of  $k^*$ -quotient  $\hat{\mathcal{F}}_{(b,G)}(Q) \cong N_G(Q, f)/C_G(Q)$  [4, 7.4].

1.4. On the other hand, for any category  $\mathfrak{C}$  and any Abelian group  $Z$  let us call *regular central  $Z$ -extension* of  $\mathfrak{C}$  any category  $\hat{\mathfrak{C}}$  over the same objects endowed with a *full* functor  $\mathfrak{c}: \hat{\mathfrak{C}} \rightarrow \mathfrak{C}$ , which is the identity over the objects, and, for any pair of  $\mathfrak{C}$ -objects  $A$  and  $B$ , with a *regular* action of  $Z$  over the *fibers* of the map

$$\hat{\mathfrak{C}}(B, A) \longrightarrow \mathfrak{C}(B, A) \quad 1.4.1$$

induced by  $\mathfrak{c}$  — where  $\mathfrak{C}(B, A)$  and  $\hat{\mathfrak{C}}(B, A)$  denote the corresponding sets of  $\mathfrak{C}$ - and  $\hat{\mathfrak{C}}$ -morphisms from  $A$  to  $B$  — in such a way that these  $Z$ -actions are compatible with the composition of  $\hat{\mathfrak{C}}$ -morphisms. Note that, if  $\mathfrak{C}'$  is a second category and  $\mathfrak{e}: \mathfrak{C} \rightarrow \mathfrak{C}'$  an equivalence of categories, we easily can obtain a *regular central  $Z$ -extension*  $\hat{\mathfrak{C}}'$  of  $\mathfrak{C}'$  and a  *$Z$ -compatible equivalence of categories*  $\hat{\mathfrak{e}}: \hat{\mathfrak{C}} \rightarrow \hat{\mathfrak{C}}'$ . In short, we call  $k^*$ -category any *regular central  $k^*$ -extension* of a category and call  $k^*$ -functor  $\hat{f}: \hat{\mathfrak{C}} \rightarrow \hat{\mathfrak{C}}'$  any functor between  $k^*$ -categories fulfilling  $\hat{f}(\lambda \cdot \hat{f}) = \lambda \cdot \hat{f}(\hat{f})$  for any  $\lambda \in k^*$  and any  $\hat{\mathfrak{C}}$ -morphism  $\hat{f}$ .

1.5. In the 2002 Durham Symposium, Markus Linckelmann [1] conjectured the existence of a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}_{(b,G)}^{\text{sc}}$  of  $\mathcal{F}_{(b,G)}^{\text{sc}}$  admitting a  $k^*$ -group isomorphism

$$\hat{\mathcal{F}}_{(b,G)}^{\text{sc}}(Q) \cong \hat{N}_G(Q, f)/C_G(Q) \quad 1.5.1$$

for any *selfcentralizing* Brauer  $(b, G)$ -pair  $(Q, f)$  contained in  $(P, e)$ . Here we show the existence of a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}$  of  $\mathcal{F}_{(b,G)}^{\text{nc}}$  admitting a  $k^*$ -group isomorphism

$$\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q) \cong \hat{N}_G(Q, f)/C_G(Q) \quad 1.5.2$$

for any *nilcentralized* Brauer  $(b, G)$ -pair  $(Q, f)$  contained in  $(P, e)$ , in particular proving Linckelmann’s conjecture above.

1.6. In both cases, these  $k^*$ -group isomorphisms are not precise enough to guarantee the uniqueness either of  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}$ , or of  $\hat{\mathcal{F}}_{(b,G)}^{\text{sc}}$  as showed in [2, Theorem 1.3]. More explicitly, if  $(Q, f)$  and  $(R, g)$  are *nilcentralized* Brauer  $(b, G)$ -pairs contained in  $(P, e)$  such that  $(R, g)$  is contained and normal in  $(Q, f)$  then, denoting by  $\hat{N}_G(Q, f)_R$  the stabilizer of  $R$  in  $\hat{N}_G(Q, f)$ , Proposition 11.23 in [4] supplies a particular  $k^*$ -group homomorphism

$$\hat{N}_G(Q, f)_R/C_G(Q) \longrightarrow \hat{N}_G(R, g)/C_G(R) \quad 1.6.1.$$

But, a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}$  of  $\mathcal{F}_{(b,G)}^{\text{nc}}$  also supplies a  $k^*$ -group homomorphism

$$\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q)_R \longrightarrow \hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(R) \quad 1.6.2,$$

where  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q)_R$  denotes the stabilizer of  $R$  in  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q)$ , sending any  $\hat{\sigma}$  in  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q)_R$  on the unique element  $\hat{\tau} \in \hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(R)$  fulfilling  $i_R^Q \circ \hat{\tau} = \hat{\sigma} \circ i_R^Q$ , where  $i_R^Q$  is a lifting to  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q, R)$  of the inclusion map  $R \subset Q$ . The *uniqueness* of a suitable *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}$  depends on the compatibility of all the  $k^*$ -group homomorphisms 1.6.1 and 1.6.2 with the corresponding  $k^*$ -group isomorphisms 1.5.2 or, more generally, it depends on the *folder structure* of  $\mathcal{F}_{(b,G)}^{\text{nc}}$  determined by [4, Theorem 11.32].

## 2. Folded Frobenius $P$ -categories

2.1. Denoting by  $P$  a finite  $p$ -group, by  $\mathbf{iGr}$  the category formed by the finite groups and by the injective group homomorphisms, and by  $\mathcal{F}_P$  the subcategory of  $\mathbf{iGr}$  where the objects are all the subgroups of  $P$  and the morphisms are the group homomorphisms induced by the conjugation by elements of  $P$ , recall that a *Frobenius  $P$ -category*  $\mathcal{F}$  is a subcategory of  $\mathbf{iGr}$  containing  $\mathcal{F}_P$  where the objects are all the subgroups of  $P$  and the morphisms fulfill the following three conditions [4, 2.8 and Proposition 2.11]

2.1.1 *If  $Q, R$  and  $T$  are subgroups of  $P$ , for any  $\varphi \in \mathcal{F}(Q, R)$  and any group homomorphism  $\psi: T \rightarrow R$ , the composition  $\varphi \circ \psi$  belongs to  $\mathcal{F}(Q, T)$  (if and) only if  $\psi \in \mathcal{F}(R, T)$ .*

2.1.2  *$\mathcal{F}_P(P)$  is a Sylow  $p$ -subgroup of  $\mathcal{F}(P)$ .*

Let us say that a subgroup  $Q$  of  $P$  is *fully centralized in  $\mathcal{F}$*  if for any  $\mathcal{F}$ -morphism  $\xi: Q \cdot C_P(Q) \rightarrow P$  we have  $\xi(C_P(Q)) = C_P(\xi(Q))$ .

2.1.3 *For any subgroup  $Q$  of  $P$  fully centralized in  $\mathcal{F}$ , any  $\mathcal{F}$ -morphism  $\varphi: Q \rightarrow P$  and any subgroup  $R$  of  $N_P(\varphi(Q))$  containing  $\varphi(Q)$  such that  $\mathcal{F}_P(Q)$  contains the action of  $\mathcal{F}_R(\varphi(Q))$  over  $Q$  via  $\varphi$ , there is an  $\mathcal{F}$ -morphism  $\zeta: R \rightarrow P$  fulfilling  $\zeta(\varphi(u)) = u$  for any  $u \in Q$ .*

2.2. With the notation in 1.1 above, it follows from [4, Theorem 3.7] that  $\mathcal{F}_{(b,G)}$  is a Frobenius  $P$ -category. Moreover, we say that a subgroup  $Q$  of  $P$  is  *$\mathcal{F}$ -nilcentralized* if, for any  $\varphi \in \mathcal{F}(P, Q)$  such that  $Q' = \varphi(Q)$  is fully centralized in  $\mathcal{F}$ , the  $C_P(Q')$ -categories  $C_{\mathcal{F}}(Q')$  [4, 2.14] and  $\mathcal{F}_{C_P(Q')}$  coincide; note that, according to [4, Proposition 7.2], in  $\mathcal{F}_{(b,G)}$  this definition agree with the definition in 1.2 above. Similarly, we say that  $Q$  is  *$\mathcal{F}$ -selfcentralizing* if

$$C_P(\varphi(Q)) \subset \varphi(Q) \quad 2.2.1$$

for any  $\varphi \in \mathcal{F}(P, Q)$ ; once again, according to [4, Corollary 7.3], in  $\mathcal{F}_{(b,G)}$  this definition agree with the definition in 1.2 above. We respectively denote by  $\mathcal{F}^{\text{nc}}$  or by  $\mathcal{F}^{\text{sc}}$  the *full* subcategories of  $\mathcal{F}$  over the respective sets of  $\mathcal{F}$ -nilcentralized or of  $\mathcal{F}$ -selfcentralizing subgroups of  $P$ .

2.3. We call  $\mathcal{F}^{\text{nc}}$ -chain any functor  $\mathbf{q} : \Delta_n \rightarrow \mathcal{F}^{\text{nc}}$  where the  $n$ -simplex  $\Delta_n$  is considered as a category where the morphisms  $—$  denoted by  $i \bullet i'$  — are defined by the order [4, A2.2]; for any  $\mathcal{F}$ -nilcentralized subgroup  $Q$  of  $P$ , let us denote by  $\mathbf{q}_Q : \Delta_0 \rightarrow \mathcal{F}^{\text{nc}}$  the obvious  $\mathcal{F}^{\text{nc}}$ -chain sending 0 to  $Q$ . Following [4, A2.8], we denote by  $\mathbf{ch}^*(\mathcal{F}^{\text{nc}})$  the category where the objects are all the  $\mathcal{F}^{\text{nc}}$ -chains  $(\mathbf{q}, \Delta_n)$  and the morphisms from  $\mathbf{q} : \Delta_n \rightarrow \mathcal{F}^{\text{nc}}$  to another  $\mathcal{F}^{\text{nc}}$ -chain  $\mathbf{r} : \Delta_m \rightarrow \mathcal{F}^{\text{nc}}$  are the pairs  $(\nu, \delta)$  formed by an *order preserving map*  $\delta : \Delta_m \rightarrow \Delta_n$  and by a *natural isomorphism*  $\nu : \mathbf{q} \circ \delta \cong \mathbf{r}$ , the composition being defined by the formula

$$(\mu, \varepsilon) \circ (\nu, \delta) = (\mu \circ (\nu * \varepsilon), \delta \circ \varepsilon) \quad 2.3.1.$$

Recall that we have a canonical functor [4, Proposition A2.10]

$$\mathbf{aut}_{\mathcal{F}^{\text{nc}}} : \mathbf{ch}^*(\mathcal{F}^{\text{nc}}) \longrightarrow \mathfrak{Gr} \quad 2.3.2$$

mapping any  $\mathcal{F}^{\text{nc}}$ -chain  $\mathbf{q} : \Delta_n \rightarrow \mathcal{F}^{\text{nc}}$  to the group of *natural automorphisms* of  $\mathbf{q}$ .

2.4. In [5, §2] we introduce a *folded Frobenius P-category*  $(\mathcal{F}, \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}})$  as a pair formed by a Frobenius  $P$ -category  $\mathcal{F}$  and a functor

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}} : \mathbf{ch}^*(\mathcal{F}^{\text{sc}}) \longrightarrow k^* - \mathfrak{Gr} \quad 2.4.1$$

lifting the canonical functor  $\mathbf{aut}_{\mathcal{F}^{\text{sc}}}$ ; here, we call *folded Frobenius P-category*  $(\mathcal{F}, \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}})$  a pair formed by  $\mathcal{F}$  and a functor

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}} : \mathbf{ch}^*(\mathcal{F}^{\text{nc}}) \longrightarrow k^* - \mathfrak{Gr} \quad 2.4.2$$

lifting the canonical functor  $\mathbf{aut}_{\mathcal{F}^{\text{nc}}}$ ; we also say that  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}}$  is a *folder structure* of  $\mathcal{F}$ . With the notation of 1.1 above, Theorem 11.32 in [4] exhibits a *folder structure* of  $\mathcal{F}_{(b,G)}$ , namely a functor  $\widehat{\mathbf{aut}}_{(\mathcal{F}_{(b,G)})^{\text{nc}}}$  lifting  $\mathbf{aut}_{(\mathcal{F}_{(b,G)})^{\text{nc}}}$ , that we call *Brauer folder structure* of  $\mathcal{F}_{(b,G)}$ . Actually, both definitions coincide since any functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}}$  lifting  $\mathbf{aut}_{\mathcal{F}^{\text{sc}}}$  can be extended to a unique functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}}$  lifting  $\mathbf{aut}_{\mathcal{F}^{\text{nc}}}$ , as it shows our next result.

**Theorem 2.5.** *Any functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}}$  lifting  $\mathbf{aut}_{\mathcal{F}^{\text{sc}}}$  to the category  $k^* - \mathfrak{Gr}$  can be extended to a unique functor lifting  $\mathbf{aut}_{\mathcal{F}^{\text{nc}}}$*

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}} : \mathbf{ch}^*(\mathcal{F}^{\text{nc}}) \longrightarrow k^* - \mathfrak{Gr} \quad 2.5.1.$$

**Proof:** Let  $\mathfrak{X}$  be a set of  $\mathcal{F}$ -nilcentralized subgroups of  $P$  which contains all the  $\mathcal{F}$ -selfcentralizing subgroups of  $P$  and is stable by  $\mathcal{F}$ -isomorphisms; denoting by  $\mathcal{F}^{\mathfrak{X}}$  the *full* subcategory of  $\mathcal{F}$  over  $\mathfrak{X}$ , we assume that  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}}$  can be extended to a unique functor

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}} : \mathbf{ch}^*(\mathcal{F}^{\mathfrak{X}}) \longrightarrow k^* - \mathfrak{Gr} \quad 2.5.2.$$

Assuming that  $\mathfrak{X}$  does not coincide with the set of all the  $\mathcal{F}$ -nilcentralized subgroups of  $P$ , let  $V$  be a maximal  $\mathcal{F}$ -nilcentralized subgroup which is not in  $\mathfrak{X}$ ; denoting by  $\mathfrak{Y}$  the union of  $\mathfrak{X}$  with all the subgroups of  $P$   $\mathcal{F}$ -isomorphic to  $V$ , it is clear that it suffices to prove that  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}$  admits a unique extension to  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{Y}})$ .

For any chain  $\mathfrak{q} : \Delta_n \rightarrow \mathcal{F}^{\mathfrak{Y}}$ , we choose an  $\mathcal{F}$ -morphism  $\alpha : \mathfrak{q}(n) \rightarrow P$  such that  $\alpha(\mathfrak{q}(n))$  is *fully centralized* in  $\mathcal{F}$  [4, Proposition 2.7] and denote by  $\mathfrak{q}^\alpha : \Delta_{n+1} \rightarrow \mathcal{F}^{\mathfrak{Y}}$  the chain mapping  $n+1$  on  $\alpha(\mathfrak{q}(n)) \cdot C_P(\alpha(\mathfrak{q}(n)))$  and  $n \bullet n+1$  on the  $\mathcal{F}$ -morphism  $\mathfrak{q}(n) \rightarrow \alpha(\mathfrak{q}(n)) \cdot C_P(\alpha(\mathfrak{q}(n)))$  induced by  $\alpha$ ; we have an obvious  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{Y}})$ -morphism [4, A3.1]

$$(\mathrm{id}_{\mathfrak{q}}, \delta_{n+1}^n) : (\mathfrak{q}^\alpha, \Delta_{n+1}) \longrightarrow (\mathfrak{q}, \Delta_n) \quad 2.5.3$$

and the functor  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{Y}}}$  maps  $(\mathrm{id}_{\mathfrak{q}}, \delta_{n+1}^n)$  on a group homomorphism

$$\mathcal{F}(\mathfrak{q}^\alpha) \longrightarrow \mathcal{F}(\mathfrak{q}) \quad 2.5.4$$

which is surjective since any  $\theta \in \mathcal{F}(\mathfrak{q}) \subset \mathcal{F}(\mathfrak{q}(n))$  can be “extended” to an  $\mathcal{F}$ -automorphism of  $\mathfrak{q}^\alpha(n+1)$  [4, statement 2.10.1].

Then, since  $\alpha(\mathfrak{q}(n))$  is *fully centralized* in  $\mathcal{F}$ , the kernel of homomorphism 2.5.4 is a  $p$ -group [4, Corollary 4.7]; moreover, since  $\mathfrak{q}^\alpha(n+1)$  belongs to  $\mathfrak{X}$ , the functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}$  and the structural inclusion  $\mathcal{F}(\mathfrak{q}^\alpha) \subset \mathcal{F}(\mathfrak{q}^\alpha(n+1))$  determine a  $k^*$ -subgroup

$$\hat{\mathcal{F}}(\mathfrak{q}^\alpha) \subset \hat{\mathcal{F}}(\mathfrak{q}^\alpha(n+1)) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}^\alpha(n+1)) \quad 2.5.5$$

and, since the kernel of homomorphism 2.5.4 is a  $p$ -group, this  $k^*$ -subgroup induces a central  $k^*$ -extension  $\hat{\mathcal{F}}(\mathfrak{q})$  of  $\mathcal{F}(\mathfrak{q})$  such that we have a surjective  $k^*$ -group homomorphism

$$\hat{\mathcal{F}}(\mathfrak{q}^\alpha) \longrightarrow \hat{\mathcal{F}}(\mathfrak{q}) \quad 2.5.6$$

lifting homomorphism 2.5.4.

Note that, for a different choice  $\alpha' : \mathfrak{q}(n) \rightarrow P$  of  $\alpha$ , we have an  $\mathcal{F}$ -isomorphism  $\alpha(\mathfrak{q}(n)) \cong \alpha'(\mathfrak{q}(n))$  which can be extended to an  $\mathcal{F}$ -isomorphism  $\mathfrak{q}^\alpha(n+1) \cong \mathfrak{q}^{\alpha'}(n+1)$  and then  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}$  determines a  $k^*$ -isomorphism

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}^\alpha(n+1)) \cong \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}^{\alpha'}(n+1)) \quad 2.5.7$$

mapping  $\hat{\mathcal{F}}(\mathfrak{q}^\alpha)$  onto  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha'})$ ; moreover, it follows from [4, Proposition 4.6] that two such  $\mathcal{F}$ -isomorphisms are  $C_P(\alpha'(\mathfrak{q}(n)))$ -conjugate and therefore

our definition of  $\hat{\mathcal{F}}(\mathfrak{q})$  does not depend on our choice of  $\alpha$ . Similarly, if  $\mathfrak{q}(n)$  belongs to  $\mathfrak{X}$  then the functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}$  already defines a  $k^*$ -group  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}(n))$  and, denoting by  $\mathfrak{q}_{n,n+1}^{\alpha} : \Delta_1 \rightarrow \mathcal{F}^{\mathfrak{X}}$  the chain mapping 0 on  $\mathfrak{q}(n)$ , 1 on  $\mathfrak{q}^{\alpha}(n+1)$  and  $0 \bullet 1$  on  $\mathfrak{q}^{\alpha}(n \bullet n+1)$ , a  $k^*$ -group homomorphism

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}_{n,n+1}^{\alpha}) \longrightarrow \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}(n)) \quad 2.5.8$$

inducing a *canonical*  $k^*$ -group isomorphism from  $\hat{\mathcal{F}}(\mathfrak{q})$  in 2.5.6 above onto the inverse image of  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{Y}}}(\mathfrak{q}) \subset \mathbf{aut}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}(n))$  in  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q}(n))$ ; in particular, if the image of  $\mathfrak{q}$  is contained in  $\mathfrak{X}$ , we get a *canonical*  $k^*$ -group isomorphism  $\hat{\mathcal{F}}(\mathfrak{q}) \cong \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}}(\mathfrak{q})$ .

Now, for any  $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{Y}})$ -morphism  $(\nu, \delta) : (\mathfrak{r}, \Delta_m) \rightarrow (\mathfrak{q}, \Delta_n)$ , choosing suitable  $\mathcal{F}$ -morphisms  $\alpha : \mathfrak{q}(n) \rightarrow P$  and  $\beta : \mathfrak{r}(m) \rightarrow P$  as above, we have to exhibit a  $k^*$ -group homomorphism  $\hat{\mathcal{F}}(\mathfrak{r}) \rightarrow \hat{\mathcal{F}}(\mathfrak{q})$  lifting  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{Y}}}(\nu, \delta)$ . Firstly, we assume that the image of  $\mathfrak{r}(\delta(n))$  *via*  $\mathfrak{r}(\delta(n) \bullet m)$  is *normal* in  $\mathfrak{r}(m)$ ; in this case,  $\beta(\mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n))))$  is normal in  $\mathfrak{r}^{\beta}(m+1)$  and, according to [4, statement 2.10.1], there is an  $\mathcal{F}$ -morphism

$$\hat{\nu} : \mathfrak{r}^{\beta}(m+1) \longrightarrow N_P(\alpha(\mathfrak{q}(n))) \quad 2.5.9$$

extending the  $\mathcal{F}$ -morphism

$$\beta(\mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n)))) \cong \mathfrak{r}(\delta(n))^{\nu_n} \cong \mathfrak{q}(n) \cong \alpha(\mathfrak{q}(n)) \subset P \quad 2.5.10,$$

and we set  $U = \hat{\nu}(\mathfrak{r}^{\beta}(m+1)) \cdot C_P(\alpha(\mathfrak{q}(n)))$ . Then, we consider the chains

$$\mathfrak{q}^{\alpha, \nu} : \Delta_{n+2} \longrightarrow \mathcal{F}^{\mathfrak{Y}} \quad \text{and} \quad \mathfrak{r}^{\beta, \nu} : \Delta_{m+2} \longrightarrow \mathcal{F}^{\mathfrak{Y}} \quad 2.5.11$$

respectively extending the chains  $\mathfrak{q}^{\alpha}$  and  $\mathfrak{r}^{\beta}$  defined above, fulfilling

$$\mathfrak{q}^{\alpha, \nu}(n+2) = U = \mathfrak{r}^{\beta, \nu}(m+2) \quad 2.5.12$$

and, since  $\alpha(\mathfrak{q}(n)) \subset \hat{\nu}(\beta(\mathfrak{r}(m)))$ , respectively mapping  $n+1 \bullet n+2$  and  $m+1 \bullet m+2$  on the inclusion  $\mathfrak{q}^{\alpha}(n+1) \subset U$  and on the  $\mathcal{F}$ -morphism from  $\mathfrak{r}^{\beta}(m+1)$  to  $U$  induced by  $\hat{\nu}$ . Note that, since the centralizer of  $\alpha(\mathfrak{q}(n))$  contains  $C_P(\hat{\nu}(\beta(\mathfrak{r}(m))))$ , we still have  $U = \hat{\nu}(\beta(\mathfrak{r}(m))) \cdot C_P(\alpha(\mathfrak{q}(n)))$ . Moreover, it follows from [4, Proposition 4.6] that another choice  $\hat{\nu}'$  of the  $\mathcal{F}$ -morphism 2.5.9 is  $C_P(\alpha(\mathfrak{q}(n)))$ -conjugate of  $\hat{\nu}$  and, in particular, the group  $U$  does not change.

With all this notation, we have obvious  $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{Y}})$ -morphisms

$$\begin{aligned} (\mathrm{id}_{\mathfrak{q}^{\alpha}}, \delta_{n+2}^{n+1}) : (\mathfrak{q}^{\alpha, \nu}, \Delta_{n+2}) &\longrightarrow (\mathfrak{q}^{\alpha}, \Delta_{n+1}) \\ (\mathrm{id}_{\mathfrak{r}^{\beta}}, \delta_{m+2}^{m+1}) : (\mathfrak{r}^{\beta, \nu}, \Delta_{m+2}) &\longrightarrow (\mathfrak{r}^{\beta}, \Delta_{m+1}) \end{aligned} \quad 2.5.13$$

and, considering the maps

$$\Delta_{n+2} \xleftarrow{\sigma_n} \Delta_1 \xrightarrow{\sigma_m} \Delta_{m+2} \quad \text{and} \quad \Delta_{n+1} \xleftarrow{\tau_n} \Delta_0 \xrightarrow{\tau_m} \Delta_{m+1} \quad 2.5.14$$

respectively mapping  $i$  on  $i+n+1$  or on  $i+m+1$ , the  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphisms above determine the following  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{x}})$ -morphisms

$$(\mathfrak{q}^{\alpha,\nu} \circ \sigma_n, \Delta_1) \longrightarrow (\mathfrak{q}^{\alpha} \circ \tau_n, \Delta_0) \quad \text{and} \quad (\mathfrak{r}^{\beta,\nu} \circ \sigma_m, \Delta_1) \longrightarrow (\mathfrak{r}^{\beta} \circ \tau_m, \Delta_0) \quad 2.5.15.$$

Then, the functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{x}}}$  maps these morphisms on  $k^*$ -group homomorphisms

$$\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu} \circ \sigma_n) \longrightarrow \hat{\mathcal{F}}(\mathfrak{q}^{\alpha} \circ \tau_n) \quad \text{and} \quad \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu} \circ \sigma_m) \longrightarrow \hat{\mathcal{F}}(\mathfrak{r}^{\beta} \circ \tau_m) \quad 2.5.16.$$

But note that  $\mathcal{F}(\mathfrak{q}^{\alpha,\nu})$ ,  $\mathcal{F}(\mathfrak{q}^{\alpha})$ ,  $\mathcal{F}(\mathfrak{r}^{\beta,\nu})$  and  $\mathcal{F}(\mathfrak{r}^{\beta})$  are respectively contained in  $\mathcal{F}(\mathfrak{q}^{\alpha,\nu} \circ \sigma_n)$ ,  $\mathcal{F}(\mathfrak{q}^{\alpha} \circ \tau_n)$ ,  $\mathcal{F}(\mathfrak{r}^{\beta,\nu} \circ \sigma_m)$  and  $\mathcal{F}(\mathfrak{r}^{\beta} \circ \tau_m)$ , and therefore, considering the corresponding inverse images in  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu} \circ \sigma_n)$ ,  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha} \circ \tau_n)$ ,  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu} \circ \sigma_m)$  and  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta} \circ \tau_m)$ , the  $k^*$ -group homomorphisms 2.5.16 induce  $k^*$ -group homomorphisms (cf. 2.5.8)

$$\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{q}^{\alpha}) \quad \text{and} \quad \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{r}^{\beta}) \quad 2.5.17.$$

More explicitly, we actually have

$$\mathcal{F}(\mathfrak{q}^{\alpha} \circ \tau_n) = \mathcal{F}(U) = \mathcal{F}(\mathfrak{r}^{\beta} \circ \tau_m) \quad 2.5.18$$

and the structural inclusions  $\mathcal{F}(\mathfrak{q}^{\alpha,\nu}) \subset \mathcal{F}(U)$  and  $\mathcal{F}(\mathfrak{r}^{\beta,\nu}) \subset \mathcal{F}(U)$  induce an inclusion  $\mathcal{F}(\mathfrak{r}^{\beta,\nu}) \subset \mathcal{F}(\mathfrak{q}^{\alpha,\nu})$ ; indeed, an element  $\theta$  in  $\mathcal{F}(\mathfrak{r}^{\beta,\nu})$  stabilizes the subgroups  $\hat{\nu}(\beta(\mathfrak{r}(i \bullet m)(\mathfrak{r}(i))))$  of  $U$  for any  $i \in \Delta_m$ , so that it stabilizes

$$\alpha(\mathfrak{q}(n)) = \hat{\nu}\left(\beta\left(\mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n)))\right)\right) \quad 2.5.19,$$

and therefore  $\theta$  also stabilizes  $C_P(\alpha(\mathfrak{q}(n))) = C_U(\alpha(\mathfrak{q}(n)))$ ; thus, it stabilizes the subgroup  $\mathfrak{q}^{\alpha}(n+1)$  of  $U$  and therefore  $\theta$  belongs to  $\mathcal{F}(\mathfrak{q}^{\alpha,\nu})$ .

Moreover, we claim that

$$(\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{r}^{\beta}}, \delta_{m+2}^{m+1}))(\mathcal{F}(\mathfrak{r}^{\beta,\nu})) = \mathcal{F}(\mathfrak{r}^{\beta}) \quad 2.5.20.$$

Indeed, an element  $\theta$  in  $\mathcal{F}(\mathfrak{r}^{\beta})$  acts on  $\beta(\mathfrak{r}(m))$  determining an automorphism  $\hat{\theta}$  of  $\hat{\nu}(\beta(\mathfrak{r}(m)))$  and, as above, this automorphism stabilizes  $\alpha(\mathfrak{q}(n))$  inducing an  $\mathcal{F}$ -morphism

$$\eta : \alpha(\mathfrak{q}(n)) \cong \alpha(\mathfrak{q}(n)) \subset P \quad 2.5.21;$$

but, we are assuming that  $\alpha(\mathfrak{q}(n))$  is normal in  $\hat{\nu}(\beta(\mathfrak{r}(m)))$ , so that this group is normal in  $\mathfrak{r}^{\beta,\nu}(m+2)$  (cf. 2.5.12). Hence, it follows from [4, statement 2.10.1] that  $\eta$  can be extended to an  $\mathcal{F}$ -morphism  $\hat{\eta} : \mathfrak{r}^{\beta,\nu}(m+2) \rightarrow P$ ; then, the restriction of  $\hat{\eta}$  to  $\hat{\nu}(\beta(\mathfrak{r}(m)))$  and the  $\mathcal{F}$ -morphism

$$\hat{\nu}(\beta(\mathfrak{r}(m))) \stackrel{\hat{\theta}}{\cong} \hat{\nu}(\beta(\mathfrak{r}(m))) \subset P \quad 2.5.22$$

coincide over the subgroup  $\alpha(\mathfrak{q}(n))$  and therefore, according to [4, Proposition 4.6], these homomorphisms are  $C_P(\alpha(\mathfrak{q}(n)))$ -conjugate. In conclusion, up to a modification in our choice of  $\hat{\eta}$ , we may assume that the restriction of  $\hat{\eta}$  to  $\hat{\nu}(\beta(\mathfrak{r}(m)))$  coincides with  $\hat{\theta}$  and therefore that  $\hat{\eta}$  stabilizes  $\hat{\nu}(\mathfrak{r}^{\beta,\nu}(m+1))$  and  $\hat{\nu}(\mathfrak{r}^{\beta,\nu}(m+2))$ , so that  $\hat{\eta}$  induces an element of  $\mathcal{F}(\mathfrak{r}^{\beta,\nu})$  lifting  $\theta$ .

Consequently, we have the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{F}(U) & \supset & \mathcal{F}(\mathfrak{q}^{\alpha,\nu}) & \longrightarrow & \mathcal{F}(\mathfrak{q}^\alpha) & \longrightarrow & \mathcal{F}(\mathfrak{q}) \\ & & \cup & & \text{aut}_{\mathcal{F}^\mathfrak{y}}(\nu, \delta) \uparrow & & \\ \mathcal{F}(U) & \supset & \mathcal{F}(\mathfrak{r}^{\beta,\nu}) & \longrightarrow & \mathcal{F}(\mathfrak{r}^\beta) & \longrightarrow & \mathcal{F}(\mathfrak{r}) \end{array} \quad 2.5.23;$$

Moreover, since  $\mathfrak{q}^\alpha(n+1)$  and  $\mathfrak{r}^\beta(m+1)$  are  $\mathcal{F}$ -selfcentralizing, the kernels of the compositions of the horizontal arrows are  $\mathcal{F}_{C_U(\alpha(\mathfrak{q}(n)))}(U)$  for the top and  $\mathcal{F}_{C_U(\hat{\nu}(\beta(\mathfrak{r}(m))))}(U)$  for the bottom, and the bottom composition is surjective; hence, since  $\mathcal{F}_{C_U(\hat{\nu}(\beta(\mathfrak{r}(m))))}(U)$  is contained in  $\mathcal{F}_{C_U(\alpha(\mathfrak{q}(n)))}(U)$  and they respectively lift canonically to  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu})$  and to  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu})$ , we get a *unique*  $k^*$ -group homomorphism

$$\widehat{\text{aut}}_{\mathcal{F}^\mathfrak{y}}(\nu, \delta) : \hat{\mathcal{F}}(\mathfrak{r}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{q}) \quad 2.5.24$$

lifting  $\text{aut}_{\mathcal{F}^\mathfrak{y}}(\nu, \delta)$  and such that the corresponding diagram of  $k^*$ -group homomorphisms

$$\begin{array}{ccccccc} \hat{\mathcal{F}}(U) & \supset & \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu}) & \longrightarrow & \hat{\mathcal{F}}(\mathfrak{q}^\alpha) & \longrightarrow & \hat{\mathcal{F}}(\mathfrak{q}) \\ & & \cup & & \widehat{\text{aut}}_{\mathcal{F}^\mathfrak{y}}(\nu, \delta) \uparrow & & \\ \hat{\mathcal{F}}(U) & \supset & \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu}) & \longrightarrow & \hat{\mathcal{F}}(\mathfrak{r}^\beta) & \longrightarrow & \hat{\mathcal{F}}(\mathfrak{r}) \end{array} \quad 2.5.25$$

is commutative.

Consider another  $\text{ch}^*(\mathcal{F}^\mathfrak{y})$ -morphism  $(\mu, \varepsilon) : (\mathfrak{t}, \Delta_\ell) \rightarrow (\mathfrak{r}, \Delta_m)$ , so that

$$(\nu, \delta) \circ (\mu, \varepsilon) = (\nu \circ (\mu * \delta), \varepsilon \circ \delta) \quad 2.5.26$$

and set  $\lambda = \nu \circ (\mu * \delta)$  and  $\varphi = \varepsilon \circ \delta$ ; then, choosing a suitable  $\mathcal{F}$ -morphism  $\gamma : \mathfrak{t}(\ell) \rightarrow P$  as above, we still assume that the images of  $\mathfrak{t}(\varphi(n))$  via  $\mathfrak{t}(\varphi(n) \bullet \ell)$



and of  $\mathfrak{t}(\varepsilon(m))$  via  $\mathfrak{t}(\varepsilon(m) \bullet \ell)$  are *normal* in  $\mathfrak{t}(\ell)$ . In particular, this implies that the image of  $\mathfrak{r}(\delta(n))$  via  $\mathfrak{r}(\delta(n) \bullet m)$  is *normal* in  $\mathfrak{r}(m)$ ; that is to say, we have already defined the  $k^*$ -group homomorphisms  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta)$ ,  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon)$  and  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi)$  respectively lifting  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta)$ ,  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon)$  and  $\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi)$  and we want to prove that

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) \quad 2.5.27.$$

More explicitly, applying the construction in 2.5.9 above to the  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphisms  $(\nu, \delta)$ ,  $(\mu, \varepsilon)$  and  $(\varphi, \lambda)$ , we get  $\mathcal{F}$ -morphisms

$$\begin{aligned} \hat{\nu} : \mathfrak{r}^{\beta}(m+1) &\longrightarrow N_P\left(\alpha(\mathfrak{q}(n))\right) \\ \hat{\mu} : \mathfrak{t}^{\gamma}(\ell+1) &\longrightarrow N_P\left(\beta(\mathfrak{r}(m))\right) \\ \hat{\lambda} : \mathfrak{t}^{\gamma}(\ell+1) &\longrightarrow N_P\left(\alpha(\mathfrak{q}(n))\right) \end{aligned} \quad 2.5.28;$$

actually, it is clear that the respective images of  $\hat{\nu}$ ,  $\hat{\mu}$  and  $\hat{\lambda}$  are respectively contained in  $\mathfrak{q}^{\alpha}(n+1)$ ,  $\mathfrak{r}^{\beta}(m+1)$  and  $\mathfrak{q}^{\alpha}(n+1)$  and, with evident notation, our construction can be explicited in the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{t}(\ell) & \cong & \gamma(\mathfrak{t}(\ell)) \subset \mathfrak{t}^{\gamma}(\ell+1) & \xrightarrow{\hat{\lambda}} & \mathfrak{q}^{\alpha}(n+1) \\ & & \parallel & & \\ & & \mathfrak{t}^{\gamma}(\ell+1) \xrightarrow{\hat{\mu}} \mathfrak{r}^{\beta}(m+1) & & \\ \uparrow & & \parallel & & \\ \mathfrak{t}(\varepsilon(m)) & \xrightarrow{\mu_m} & \mathfrak{r}(m) \cong \beta(\mathfrak{r}(m)) \subset \mathfrak{r}^{\beta}(m+1) & \xrightarrow{\hat{\nu}} & \mathfrak{q}^{\alpha}(n+1) \\ \uparrow & & \uparrow & & \cup \\ \mathfrak{t}(\varphi(n)) & \xrightarrow{\mu_{\delta(n)}} & \mathfrak{r}(\delta(n)) \xrightarrow{\nu_n} \mathfrak{q}(n) & \cong & \alpha(\mathfrak{q}(n)) \end{array} \quad 2.5.29.$$

That is to say, according to 2.5.10 above,  $\hat{\lambda}$ ,  $\hat{\mu}$  and  $\hat{\nu}$  respectively extend the  $\mathcal{F}$ -morphisms

$$\begin{aligned} \gamma\left(\mathfrak{t}(\varphi(n) \bullet \ell)(\mathfrak{t}(\varphi(n)))\right) &\cong \mathfrak{t}(\varphi(n)) \xrightarrow{\lambda_n} \mathfrak{q}(n) \cong \alpha(\mathfrak{q}(n)) \subset P \\ \gamma\left(\mathfrak{t}(\varepsilon(m) \bullet \ell)(\mathfrak{t}(\varepsilon(m)))\right) &\cong \mathfrak{t}(\varepsilon(m)) \xrightarrow{\mu_m} \mathfrak{r}(m) \cong \beta(\mathfrak{r}(m)) \subset P \\ \beta\left(\mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n)))\right) &\cong \mathfrak{r}(\delta(n)) \xrightarrow{\nu_n} \mathfrak{q}(n) \cong \alpha(\mathfrak{q}(n)) \subset P \end{aligned} \quad 2.5.30$$

and, since  $\beta\left(\mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n)))\right)$  is contained in  $\beta(\mathfrak{r}(m))$ , it is easily checked that the composition  $\hat{\nu} \circ \hat{\mu}$  also extends the top  $\mathcal{F}$ -morphism in 2.5.30; then, as above, it follows again from [4, Proposition 4.6] that  $\hat{\lambda}$  and  $\hat{\nu} \circ \hat{\mu}$  are  $C_P\left(\alpha(\mathfrak{q}(n))\right)$ -conjugate; actually, up to a modification of our choice of  $\hat{\lambda}$ , we may assume that they coincide.

Note that, as in 2.5.17 above, we get the  $k^*$ -group homomorphisms

$$\begin{aligned} \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu}) &\longrightarrow \hat{\mathcal{F}}(\mathfrak{r}^\beta) & \text{and} & & \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu}) &\longrightarrow \hat{\mathcal{F}}(\mathfrak{t}^\gamma) \\ \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}) &\longrightarrow \hat{\mathcal{F}}(\mathfrak{q}^\alpha) & \text{and} & & \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda}) &\longrightarrow \hat{\mathcal{F}}(\mathfrak{t}^\gamma) \end{aligned} \quad 2.5.31;$$

similarly, we have the inclusions

$$\hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu}) \subset \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu}) \quad \text{and} \quad \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda}) \subset \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}) \quad 2.5.32$$

and, as in 2.5.20, we obtain

$$\begin{aligned} (\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{t}^\gamma}, \delta_{\ell+2}^{\ell+1}))(\mathcal{F}(\mathfrak{t}^{\gamma,\mu})) &= \mathcal{F}(\mathfrak{t}^\gamma) \\ (\mathbf{aut}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{t}^\gamma}, \delta_{\ell+2}^{\ell+1}))(\mathcal{F}(\mathfrak{t}^{\gamma,\lambda})) &= \mathcal{F}(\mathfrak{t}^\gamma) \end{aligned} \quad 2.5.33.$$

Moreover, we have to consider chains

$$\begin{aligned} \mathfrak{q}^{\alpha,\nu,\lambda} : \Delta_{n+3} &\longrightarrow \mathcal{F}^{\mathfrak{y}} \\ \mathfrak{r}^{\beta,\mu,\lambda} : \Delta_{m+3} &\longrightarrow \mathcal{F}^{\mathfrak{y}} \\ \mathfrak{t}^{\gamma,\mu,\lambda} : \Delta_{\ell+3} &\longrightarrow \mathcal{F}^{\mathfrak{y}} \end{aligned} \quad 2.5.34$$

respectively extending the chains  $\mathfrak{q}^{\alpha,\nu}$ ,  $\mathfrak{r}^{\beta,\mu}$  and  $\mathfrak{t}^{\gamma,\mu}$ ; recall that (cf. 2.5.12)

$$\begin{aligned} \mathfrak{q}^{\alpha,\nu}(n+2) &= \hat{\nu}(\beta(\mathfrak{r}(m))) \cdot C_P(\alpha(\mathfrak{q}(n))) \\ \mathfrak{r}^{\beta,\mu}(m+2) &= \hat{\mu}(\gamma(\mathfrak{t}(\ell))) \cdot C_P(\beta(\mathfrak{r}(m))) = \mathfrak{t}^{\gamma,\mu}(\ell+2) \end{aligned} \quad 2.5.35$$

and that, according to our remark above and since we assume that  $\hat{\lambda} = \hat{\nu} \circ \hat{\mu}$ , we still have

$$\mathfrak{q}^{\alpha,\lambda}(n+2) = \hat{\nu}(\hat{\mu}(\gamma(\mathfrak{t}(\ell)))) \cdot C_P(\alpha(\mathfrak{q}(n))) \quad 2.5.36;$$

thus, since  $\beta(\mathfrak{r}(m)) \subset \hat{\mu}(\gamma(\mathfrak{t}(\ell)))$ , we get  $\mathfrak{q}^{\alpha,\nu}(n+2) \subset \mathfrak{q}^{\alpha,\lambda}(n+2)$  and, since the centralizer of  $\alpha(\mathfrak{q}(n))$  contains the centralizer of  $\hat{\nu}(\beta(\mathfrak{r}(m)))$ ,  $\hat{\nu}$  induces an  $\mathcal{F}$ -morphism

$$\mathfrak{r}^{\beta,\mu}(m+2) = \mathfrak{t}^{\gamma,\mu}(\ell+2) \longrightarrow \mathfrak{q}^{\alpha,\lambda}(n+2) \quad 2.5.37;$$

then, we complete our definition of  $\mathfrak{q}^{\alpha,\nu,\lambda}$ ,  $\mathfrak{r}^{\beta,\mu,\lambda}$  and  $\mathfrak{t}^{\gamma,\mu,\lambda}$  by setting

$$\mathfrak{q}^{\alpha,\nu,\lambda}(n+3) = \mathfrak{r}^{\beta,\mu,\lambda}(m+3) = \mathfrak{t}^{\gamma,\mu,\lambda}(\ell+3) = \mathfrak{q}^{\alpha,\lambda}(n+2) \quad 2.5.38,$$

and respectively mapping  $n+2 \bullet n+3$ ,  $m+2 \bullet m+3$  and  $\ell+2 \bullet \ell+3$  on the inclusion  $\mathfrak{q}^{\alpha,\nu}(n+2) \subset \mathfrak{q}^{\alpha,\lambda}(n+2)$  and on the  $\mathcal{F}$ -morphism 2.5.34 induced by  $\hat{\nu}$ .

Note that, since we have  $\mathfrak{r}^{\beta,\mu,\lambda}(m+3) = \mathfrak{t}^{\gamma,\mu,\lambda}(\ell+3)$  and

$$\mathfrak{r}^{\beta,\mu,\lambda}(m+2 \bullet m+3) = \mathfrak{t}^{\gamma,\mu,\lambda}(\ell+2 \bullet \ell+3) \quad 2.5.39,$$

it is quite clear that, up to suitable identifications, the left-hand inclusion in 2.5.32 forces in  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}(n+2))$  a  $k^*$ -group inclusion  $\hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu,\lambda}) \subset \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu,\lambda})$ . Moreover, we claim that  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu,\lambda}) \subset \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda})$ ; indeed, up to our identifications both are  $k^*$ -subgroups of  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}(n+2))$ ; but, any  $\theta$  in  $\mathcal{F}(\mathfrak{r}^{\beta,\mu,\lambda})$  clearly stabilizes (cf. 2.5.19)

$$\begin{aligned} \mathfrak{r}^{\beta,\mu,\lambda}(\delta(n) \bullet m+3)(\mathfrak{r}(\delta(n))) &= \hat{\nu} \left( \beta \left( \mathfrak{r}(\delta(n) \bullet m)(\mathfrak{r}(\delta(n))) \right) \right) \\ &= \alpha(\mathfrak{q}(n)) \end{aligned} \quad 2.5.40,$$

so that it stabilizes  $\mathfrak{q}^{\alpha}(n+1)$  in  $\mathfrak{q}^{\alpha,\lambda}(n+2)$ ; then, it is easily checked that  $\theta$  belongs to  $\mathcal{F}(\mathfrak{q}^{\alpha,\lambda})$ .

Now, as in 2.5.16 above, considering the maps

$$\begin{aligned} \hat{\sigma}_n : \Delta_1 &\longrightarrow \Delta_{n+3} & \text{and} & & \hat{\tau}_n : \Delta_0 &\longrightarrow \Delta_{n+2} \\ \hat{\sigma}_m : \Delta_1 &\longrightarrow \Delta_{m+3} & \text{and} & & \hat{\tau}_m : \Delta_0 &\longrightarrow \Delta_{m+2} \\ \hat{\sigma}_\ell : \Delta_1 &\longrightarrow \Delta_{\ell+3} & \text{and} & & \hat{\tau}_\ell : \Delta_0 &\longrightarrow \Delta_{\ell+2} \end{aligned} \quad 2.5.41$$

respectively sending  $i$  to  $i+n+2$ , to  $i+m+2$  or to  $i+\ell+2$ , and applying the analogous argument to the obvious  $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{A}})$ -morphisms

$$\begin{aligned} (\text{id}_{\mathfrak{q}^{\alpha,\nu}}, \delta_{n+3}^{n+2}) : (\mathfrak{q}^{\alpha,\nu,\lambda}, \Delta_{n+3}) &\longrightarrow (\mathfrak{q}^{\alpha,\nu}, \Delta_{n+2}) \\ (\text{id}_{\mathfrak{q}^{\alpha,\lambda}}, \delta_{n+2}^{n+2}) : (\mathfrak{q}^{\alpha,\nu,\lambda}, \Delta_{n+3}) &\longrightarrow (\mathfrak{q}^{\alpha,\lambda}, \Delta_{n+2}) \\ (\text{id}_{\mathfrak{r}^{\beta,\mu}}, \delta_{m+3}^{m+2}) : (\mathfrak{r}^{\beta,\mu,\lambda}, \Delta_{m+3}) &\longrightarrow (\mathfrak{r}^{\beta,\mu}, \Delta_{m+2}) \\ (\text{id}_{\mathfrak{t}^{\gamma,\mu}}, \delta_{m+3}^{m+2}) : (\mathfrak{t}^{\gamma,\mu,\lambda}, \Delta_{\ell+3}) &\longrightarrow (\mathfrak{t}^{\gamma,\mu}, \Delta_{\ell+2}) \\ (\text{id}_{\mathfrak{t}^{\gamma,\mu}}, \delta_{m+2}^{m+2}) : (\mathfrak{t}^{\gamma,\mu,\lambda}, \Delta_{\ell+3}) &\longrightarrow (\mathfrak{t}^{\gamma,\lambda}, \Delta_{\ell+2}) \end{aligned} \quad 2.5.42,$$

it is clear that the functor  $\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathfrak{A}}}$  still induces  $k^*$ -group homomorphisms

$$\begin{aligned} \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}) &\longleftarrow \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu,\lambda}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu}) \\ &\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu,\lambda}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu}) \\ \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda}) &\longleftarrow \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu,\lambda}) \longrightarrow \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu}) \end{aligned} \quad 2.5.43$$

and, as above, it can be proved that all the right arrows are surjective. Actually, since  $\mathfrak{t}^{\gamma,\lambda}(\ell+2) = \mathfrak{q}^{\alpha,\lambda}(n+2)$ , it follows from equality 2.5.35 that

the left arrows in the top and in the bottom are injective; more precisely, we claim that, up to suitable identifications, we have an equality

$$\hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda}) = \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu,\lambda}) \quad 2.5.44;$$

indeed, as above, any element  $\theta \in \mathcal{F}(\mathfrak{t}^{\gamma,\lambda})$  stabilizes  $\hat{\lambda}\left(\gamma(\mathfrak{t}(i \bullet \ell)(\mathfrak{t}(i)))\right)$  for any  $i \in \Delta_\ell$ , so that it stabilizes both  $\hat{\nu}\left(\hat{\mu}\left(\gamma(\mathfrak{t}(\ell))\right)\right)$  and (cf. 2.5.30)

$$\hat{\nu}\left(\hat{\mu}\left(\gamma(\mathfrak{t}(\varepsilon(m) \bullet \ell)(\mathfrak{t}(\varepsilon(m))))\right)\right) = \hat{\nu}\left(\beta(\mathfrak{r}(m))\right) \quad 2.5.45,$$

and therefore, since  $C_P\left(\beta(\mathfrak{r}(m))\right)$  coincides with the centralizer of  $\beta(\mathfrak{r}(m))$  in  $\mathfrak{q}^{\alpha,\lambda}(n+2)$ ,  $\theta$  also stabilizes

$$\hat{\nu}\left(\mathfrak{t}^{\gamma,\mu}(\ell+2)\right) = \hat{\nu}\left(\hat{\mu}\left(\gamma(\mathfrak{t}(\ell))\right) \cdot C_P\left(\beta(\mathfrak{r}(m))\right)\right) \quad 2.5.46;$$

hence,  $\theta$  belongs to  $\mathcal{F}(\mathfrak{t}^{\gamma,\mu,\lambda})$ .

Finally, we have the following diagram of  $k^*$ -groups

$$\begin{array}{ccccccc} \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda}) & = & \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu,\lambda}) & \subset & \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu,\lambda}) & \subset & \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda}) \\ \downarrow & & \downarrow & & \downarrow & & \cup \\ \hat{\mathcal{F}}(\mathfrak{t}^\gamma) & \leftarrow & \hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\mu}) & \subset & \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu}) & & \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu,\lambda}) \\ \downarrow & & & & \downarrow & & \downarrow \\ \hat{\mathcal{F}}(\mathfrak{t}) & & & & \hat{\mathcal{F}}(\mathfrak{r}^\beta) & \leftarrow & \hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu}) \subset \hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu}) \\ & & \searrow & & \downarrow & & \downarrow \\ & & & & \hat{\mathcal{F}}(\mathfrak{r}) & & \hat{\mathcal{F}}(\mathfrak{q}^\alpha) \\ & & & & & \searrow & \downarrow \\ & & & & & & \hat{\mathcal{F}}(\mathfrak{q}) \end{array} \quad 2.5.47,$$

including in the bottom our definitions of  $\widehat{\mathbf{aut}}_{\mathcal{F}^\mathfrak{y}}(\mu, \varepsilon)$  and of  $\widehat{\mathbf{aut}}_{\mathcal{F}^\mathfrak{y}}(\nu, \delta)$ ; the functoriality of  $\widehat{\mathbf{aut}}_{\mathcal{F}^\mathfrak{x}}$  guarantees the commutativity of the top left-hand squares. Moreover, any element  $\hat{\theta}$  in  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\mu,\lambda})$  determines an element  $\hat{\theta}^\lambda$  of  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda})$  which also stabilizes  $\mathfrak{q}^{\alpha,\nu}(n+2)$  (cf. 2.5.35) in  $\mathfrak{q}^{\alpha,\lambda}(n+2)$  since it stabilizes both  $\alpha(\mathfrak{q}(n))$  (cf. 2.5.40) and

$$\mathfrak{r}^{\beta,\mu,\lambda}(m \bullet m + 3)(\mathfrak{r}(m)) = \hat{\nu}\left(\beta(\mathfrak{r}(m))\right) \quad 2.5.48;$$

hence, up to our identifications,  $\hat{\theta}^\lambda$  belongs to  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu,\lambda})$  and induces an element  $\hat{\theta}^\nu$  in  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu})$  which still stabilizes  $\mathfrak{q}^{\alpha,\nu}(n+2) = \mathfrak{r}^{\beta,\nu}(m+2)$  (cf. 2.5.12);

thus, up to our identifications, this element belongs to  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu})$  and it is easily checked that its image in  $\hat{\mathcal{F}}(\mathfrak{r}^\beta)$  coincides with the image of  $\hat{\theta}$ .

At this point, since the top vertical left-hand arrow is surjective, any element  $\hat{\eta} \in \hat{\mathcal{F}}(\mathfrak{t})$  can be lifted to some  $\hat{\eta}^{\gamma,\lambda}$  in  $\hat{\mathcal{F}}(\mathfrak{t}^{\gamma,\lambda})$  which determines elements  $\hat{\eta}^{\beta,\mu,\lambda}$  in  $\hat{\mathcal{F}}(\mathfrak{t}^{\beta,\mu,\lambda})$  and  $\hat{\eta}^{\alpha,\lambda}$  in  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\lambda})$ . Then, according to our definition, the image of  $\hat{\eta}^{\beta,\mu,\lambda}$  in  $\hat{\mathcal{F}}(\mathfrak{r})$  is just  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon)(\hat{\eta})$ ; but, we already know that  $\hat{\eta}^{\alpha,\lambda}$  determines an element  $\hat{\eta}^{\alpha,\nu}$  in  $\hat{\mathcal{F}}(\mathfrak{q}^{\alpha,\nu})$  and, once again by the functoriality of  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{x}}}$ , the image of  $\hat{\eta}^{\alpha,\nu}$  in  $\hat{\mathcal{F}}(\mathfrak{q}^\alpha)$  coincides with the image of  $\hat{\eta}^{\alpha,\lambda}$  via the bottom left-hand arrow in 2.5.31; thus, according to our definition, the image of  $\hat{\eta}^{\alpha,\nu}$  in  $\hat{\mathcal{F}}(\mathfrak{q})$  is just  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi)(\hat{\eta})$ . Finally, we already know that the image of  $\hat{\eta}^{\beta,\mu,\lambda}$  in  $\hat{\mathcal{F}}(\mathfrak{r}^\beta)$  can be lifted to the element of  $\hat{\mathcal{F}}(\mathfrak{r}^{\beta,\nu})$  determined by  $\hat{\eta}^{\alpha,\nu}$ ; consequently, according to our definition, we get

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta)(\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon)(\hat{\eta})) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi)(\hat{\eta}) \quad 2.5.49,$$

which proves equality 2.5.27.

Secondly, in the  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism  $(\nu, \delta): (\mathfrak{r}, \Delta_m) \rightarrow (\mathfrak{q}, \Delta_n)$ , assume that the image of  $\mathfrak{r}(\delta(n))$  via  $\mathfrak{r}(\delta(n) \bullet m)$  is not normal in  $\mathfrak{r}(m)$ ; let  $m'$  be the maximal element in  $\Delta_m - \Delta_{\delta(n)-1}$  (where  $\Delta_{-1} = \emptyset$ ) such that the image of  $\mathfrak{r}(\delta(n))$  via  $\mathfrak{r}(\delta(n) \bullet m')$  is normal in  $\mathfrak{r}(m')$  and, setting  $\hat{m}' = m' + 1$ , denote by  $R$  the normalizer of the image of  $\mathfrak{r}(\delta(n))$  in  $\mathfrak{r}(\hat{m}')$ , denote by  $\hat{\mathfrak{t}}: \Delta_{\hat{m}} \rightarrow \mathcal{F}^{\mathfrak{y}}$  the functor fulfilling

$$\hat{\mathfrak{t}} \circ \delta_{\hat{m}'}^m = \mathfrak{r} \quad \text{and} \quad \hat{\mathfrak{t}}(\hat{m}') = R \quad 2.5.50$$

and mapping  $\hat{m}' \bullet \hat{m}' + 1$  on the inclusion map  $R \subset \mathfrak{r}(\hat{m}')$ , and denote by  $\hat{\mathfrak{r}}'$  the restriction of  $\hat{\mathfrak{t}}$  to  $\Delta_{\hat{m}'}$ ; then, it is quite clear that  $\mathcal{F}(\hat{\mathfrak{t}}) = \mathcal{F}(\mathfrak{r})$  and it follows easily from 2.5.25 that  $\hat{\mathcal{F}}(\hat{\mathfrak{t}}) = \hat{\mathcal{F}}(\mathfrak{r})$ . Now, denoting by  $\delta': \Delta_n \rightarrow \Delta_{\hat{m}'}$  the restriction of  $\delta$ , we have an evident  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism

$$(\nu', \delta'): (\hat{\mathfrak{t}}', \Delta_{\hat{m}'}) \longrightarrow (\mathfrak{q}, \Delta_n) \quad 2.5.51$$

such that

$$(\nu', \delta') \circ (\text{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{m}'}^{\hat{m}}) = (\nu, \delta) \circ (\text{id}_{\mathfrak{r}}, \delta_{\hat{m}}^m) \quad 2.5.52$$

where  $\iota_{\hat{m}'}^{\hat{m}}: \Delta_{\hat{m}'} \rightarrow \Delta_{\hat{m}}$  denotes the natural inclusion; moreover, in 2.5.25 we have already defined both  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\text{id}_{\mathfrak{r}}, \delta_{\hat{m}}^m) = \text{id}_{\hat{\mathcal{F}}(\mathfrak{r})}$  and  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta')$ . On the other hand, note that if  $|\mathfrak{r}(m)| \neq |\mathfrak{q}(n)|$  then we have

$$|\hat{\mathfrak{t}}(\hat{m})|/|\hat{\mathfrak{t}}'(\hat{m}')| < |\mathfrak{r}(m)|/|\mathfrak{q}(n)| \quad 2.5.53$$

and therefore, arguing by induction on  $|\mathfrak{r}(m)|/|\mathfrak{q}(n)|$ , we may assume that  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\text{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{m}'}^{\hat{m}})$  is already defined. In conclusion, we define

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\text{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{m}'}^{\hat{m}}) \quad 2.5.54.$$

At present, for any pair of  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphisms  $(\nu, \delta) : (\mathfrak{r}, \Delta_m) \rightarrow (\mathfrak{q}, \Delta_n)$  and  $(\mu, \varepsilon) : (\mathfrak{t}, \Delta_\ell) \rightarrow (\mathfrak{r}, \Delta_m)$ , setting  $(\lambda, \varphi) = (\nu, \delta) \circ (\mu, \varepsilon)$ , we have already defined  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta)$ ,  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon)$  and  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi)$ , and then it remains to prove that

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) \quad 2.5.55;$$

we argue by induction first on  $|\mathfrak{t}(\ell)|/|\mathfrak{q}(n)|$  and after on  $|\mathfrak{t}(\ell)|/|\mathfrak{r}(m)|$ .

First of all, we assume that the image of  $\mathfrak{r}(\delta(n))$  *via*  $\mathfrak{r}(\delta(n) \bullet m)$  is not normal in  $\mathfrak{r}(m)$  and denote by  $m'$  the maximal element in  $\Delta_m - \Delta_{\delta(n)-1}$  such that the image of  $\mathfrak{r}(\delta(n))$  *via*  $\mathfrak{r}(\delta(n) \bullet m')$  is normal in  $\mathfrak{r}(m')$ ; moreover, denote by  $\ell'$  the maximal element in  $\Delta_\ell - \Delta_{\varphi(n)-1}$  such that the image of  $\mathfrak{t}(\varphi(n))$  *via*  $\mathfrak{t}(\varphi(n) \bullet \ell')$  is normal in  $\mathfrak{t}(\ell')$ ; then, clearly  $\varepsilon(m') \leq \ell' < \varepsilon(m)$  and, respectively denoting by  $R$  and  $T$  the normalizers of the images of  $\mathfrak{r}(\delta(n))$  in  $\mathfrak{r}(\hat{m}')$  and of  $\mathfrak{t}(\varphi(n))$  in  $\mathfrak{t}(\hat{\ell}')$ , it is clear that the isomorphism

$$\mu_{\hat{m}'} : \mathfrak{t}(\varepsilon(\hat{m}')) \cong \mathfrak{r}(\hat{m}') \quad 2.5.56$$

induces an isomorphism from  $\mathfrak{t}(\hat{\ell}' \bullet \varepsilon(\hat{m}'))(T)$  onto  $R$ . At this point, it is easily checked that we can define a  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism

$$(\hat{\mu}, \hat{\varepsilon}) : (\hat{\mathfrak{t}}, \Delta_{\hat{\ell}}) \longrightarrow (\hat{\mathfrak{r}}, \Delta_{\hat{m}}) \quad 2.5.57$$

such that

$$(\mathrm{id}_{\mathfrak{r}}, \delta_{\hat{m}'}^m) \circ (\hat{\mu}, \hat{\varepsilon}) = (\mu, \varepsilon) \circ (\mathrm{id}_{\mathfrak{t}}, \delta_{\hat{\ell}'}^\ell) \quad 2.5.58,$$

that  $\hat{\varepsilon}(\hat{m}') = \hat{\ell}'$ , that  $\hat{t}(\hat{\ell}') = T$ , that  $\hat{r}(\hat{m}') = R$  and that  $\hat{\mu}_{\hat{m}'} : T \cong R$  is determined by both  $\hat{\mu}_{\hat{m}'}$  and  $\mathfrak{t}(\hat{\ell}' \bullet \varepsilon(\hat{m}'))$ ; moreover, we consider the corresponding restriction

$$(\hat{\mu}', \hat{\varepsilon}') : (\hat{\mathfrak{t}}', \Delta_{\hat{\ell}'}) \longrightarrow (\hat{\mathfrak{r}}', \Delta_{\hat{m}'}) \quad 2.5.59$$

which obviously fulfills

$$(\mathrm{id}_{\hat{\mathfrak{r}}'}, \iota_{\hat{m}'}^{\hat{m}}) \circ (\hat{\mu}, \hat{\varepsilon}) = (\hat{\mu}', \hat{\varepsilon}') \circ (\mathrm{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{\ell}'}^{\hat{\ell}}) \quad 2.5.60.$$

Now, denoting by  $\varphi' : \Delta_n \rightarrow \Delta_{\hat{\ell}'}$  the restriction of  $\varphi$ , it is quite clear that  $\varphi' = \delta' \circ \varepsilon'$  and therefore, with the notation in 2.5.51 above, the corresponding  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism  $(\lambda', \varphi') : (\mathfrak{t}', \Delta_{\hat{\ell}'}) \longrightarrow (\mathfrak{q}, \Delta_n)$  fulfills

$$(\lambda', \varphi') = (\nu', \delta') \circ (\hat{\mu}', \hat{\varepsilon}') \quad 2.5.61$$

since we have (cf. 2.5.52, 2.5.58 and 2.5.60)

$$\begin{aligned} (\nu', \delta') \circ (\hat{\mu}', \hat{\varepsilon}') \circ (\mathrm{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{\ell}'}^{\hat{\ell}}) &= (\nu', \delta') \circ (\mathrm{id}_{\hat{\mathfrak{t}}'}, \iota_{\hat{m}'}^{\hat{m}}) \circ (\hat{\mu}, \hat{\varepsilon}) \\ &= (\nu, \delta) \circ (\mathrm{id}_{\mathfrak{r}}, \delta_{\hat{m}'}^m) \circ (\hat{\mu}, \hat{\varepsilon}) \\ &= (\nu, \delta) \circ (\mu, \varepsilon) \circ (\mathrm{id}_{\mathfrak{t}}, \delta_{\hat{\ell}'}^\ell) \\ &= (\lambda, \varphi) \circ (\mathrm{id}_{\mathfrak{t}}, \delta_{\hat{\ell}'}^\ell) \end{aligned} \quad 2.5.62;$$

hence, by the very definition 2.5.25, we have

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\nu', \delta') \circ (\hat{\mu}', \hat{\varepsilon}')) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\hat{\nu}'}, \iota_{\hat{\ell}'}^{\hat{\ell}}) \quad 2.5.63.$$

But, since  $|R|/|\mathbf{q}(n)| < |\mathbf{t}(\ell)|/|\mathbf{q}(n)|$ , it follows from the induction hypothesis that

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{nc}}}(\lambda', \varphi') = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\hat{\mu}', \hat{\varepsilon}') \quad 2.5.64;$$

similarly, since  $|\hat{\mathbf{t}}(\hat{\ell})|/|\mathbf{r}(m)| < |\mathbf{t}(\ell)|/|\mathbf{q}(n)|$ , from the induction hypothesis we get

$$\begin{aligned} \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\hat{\mu}, \hat{\varepsilon}) &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\mathrm{id}_{\mathbf{r}}, \iota_m^{\hat{m}}) \circ (\hat{\mu}, \hat{\varepsilon})) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\mu, \varepsilon) \circ (\mathrm{id}_{\mathbf{t}}, \iota_{\hat{\ell}}^{\hat{\ell}})) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) \end{aligned} \quad 2.5.65;$$

finally, since  $|\mathbf{t}(\ell)|/|R| < |\mathbf{t}(\ell)|/|\mathbf{q}(n)|$ , we still get (cf. 2.5.54 and 2.5.60)

$$\begin{aligned} \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\hat{\mu}', \hat{\varepsilon}') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\hat{\nu}'}, \iota_{\hat{\ell}'}^{\hat{\ell}}) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\mathrm{id}_{\hat{\nu}'}, \iota_{\hat{m}'}^{\hat{m}}) \circ (\hat{\mu}, \hat{\varepsilon})) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu', \delta') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\hat{\nu}'}, \iota_{\hat{m}'}^{\hat{m}}) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\hat{\mu}, \hat{\varepsilon}) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) \end{aligned} \quad 2.5.66.$$

Thus, we may assume that the image of  $\mathbf{r}(\delta(n))$  via  $\mathbf{r}(\delta(n) \bullet m)$  is normal in  $\mathbf{r}(m)$ , so that the image of  $\mathbf{t}(\varphi(n))$  via  $\mathbf{t}(\varphi(n) \bullet \varepsilon(m))$  is in particular normal in  $\mathbf{t}(\varepsilon(m))$ . As above, denote by  $\ell'$  the maximal element in  $\Delta_{\ell} - \Delta_{\varphi(n)-1}$  such that the image of  $\mathbf{t}(\varphi(n))$  via  $\mathbf{t}(\varphi(n) \bullet \ell')$  is normal in  $\mathbf{t}(\ell')$ , so that this time we have  $\varepsilon(m) \leq \ell'$ . If  $\ell' = \ell$ , it follows from 2.5.27 that we may assume that the image of  $\mathbf{t}(\varepsilon(m))$  is not normal in  $\mathbf{t}(\ell)$ ; then, *mutatis mutandi* denote by  $\ell''$  the maximal element in  $\Delta_{\ell} - \Delta_{\varepsilon(m)-1}$  such that the image of  $\mathbf{t}(\varepsilon(m))$  via  $\mathbf{t}(\varepsilon(m) \bullet \ell'')$  is normal in  $\mathbf{t}(\ell'')$ , denote by  $T'$  the normalizer of this image in  $\mathbf{t}(\ell'')$ , denote by  $\hat{\mathbf{t}}: \Delta_{\hat{\ell}} \rightarrow \mathcal{F}^{\mathfrak{y}}$  the functor fulfilling

$$\hat{\mathbf{t}} \circ \delta_{\hat{\ell}''}^{\ell} = \mathbf{t} \quad \text{and} \quad \hat{\mathbf{t}}(\hat{\ell}'') = T' \quad 2.5.67$$

and mapping  $\hat{\ell}'' \bullet \ell'' + 1$  on the inclusion map  $T' \subset \mathbf{t}(\hat{\ell}'')$ , denote by  $\hat{\mathbf{t}}'$  the restriction of  $\hat{\mathbf{t}}$  to  $\Delta_{\hat{\ell}''}$ , and denote by  $\varepsilon'': \Delta_m \rightarrow \Delta_{\hat{\ell}''}$  the restriction of  $\varepsilon$ ; as above, we have an evident  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism

$$(\mu'', \varepsilon'') : (\hat{\mathbf{t}}', \Delta_{\hat{\ell}''}) \longrightarrow (\mathbf{r}, \Delta_m) \quad 2.5.68$$

such that

$$(\mu'', \varepsilon'') \circ (\mathrm{id}_{\hat{\mathbf{t}}'}, \iota_{\hat{\ell}''}^{\hat{\ell}}) = (\mu, \varepsilon) \circ (\mathrm{id}_{\mathbf{t}}, \delta_{\hat{\ell}}^{\ell}) \quad 2.5.69.$$

Once again, it follows from our definition in 2.5.54 that we have

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu'', \varepsilon'') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{z}'}, \ell_{\hat{\ell}''}) \quad 2.5.70;$$

but according to equality 2.5.27, setting  $(\lambda'', \varphi'') = (\nu, \delta) \circ (\mu'', \varepsilon'')$ , we have

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu'', \varepsilon'') = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda'', \varphi'') \quad 2.5.71$$

and note that  $(\lambda'', \varphi'') \circ (\mathrm{id}_{\mathfrak{z}'}, \ell_{\hat{\ell}''}) = (\lambda, \varphi)$ ; moreover, the first induction index in the two compositions of  $(\nu, \delta) \circ (\mu, \varepsilon)$  and  $(\lambda'', \varphi'') \circ (\mathrm{id}_{\mathfrak{z}'}, \ell_{\hat{\ell}''})$  coincide with each other, whereas for the second one we have  $|\mathfrak{t}(\ell)|/|\mathfrak{r}(m)| > |\mathfrak{t}(\ell)|/|\mathfrak{t}(\hat{\ell}'')|$ ; consequently, it follows from our induction hypothesis that

$$\begin{aligned} \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu'', \varepsilon'') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{z}'}, \ell_{\hat{\ell}''}) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\nu, \delta) \circ (\mu'', \varepsilon'')) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{z}'}, \ell_{\hat{\ell}''}) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}((\nu, \delta) \circ (\mu, \varepsilon)) \end{aligned} \quad 2.5.72.$$

If  $\ell' \neq \ell$ , denoting by  $\varepsilon': \Delta_m \rightarrow \Delta_{\hat{\ell}'}$  the restriction of  $\varepsilon$ , with the notation in 2.5.59 above, as in 2.5.51 we have a  $\mathbf{ch}^*(\mathcal{F}^{\mathfrak{y}})$ -morphism

$$(\mu', \varepsilon') : (\mathfrak{t}', \Delta_{\hat{\ell}'}) \longrightarrow (\mathfrak{r}, \Delta_m) \quad 2.5.73$$

fulfilling (cf. 2.5.52)

$$(\mu', \varepsilon') \circ (\mathrm{id}_{\mathfrak{t}'}, \ell_{\hat{\ell}'}) = (\mu, \varepsilon) \circ (\mathrm{id}_{\mathfrak{t}}, \delta_{\hat{\ell}}^{\ell}) \quad 2.5.74;$$

hence, it follows from our definition in 2.5.54 that

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu', \varepsilon') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{t}'}, \ell_{\hat{\ell}'}) \quad 2.5.75.$$

Moreover, it is clear that  $\varphi' = \delta \circ \varepsilon'$  and therefore we get

$$(\lambda', \varphi') = (\nu, \delta) \circ (\mu', \varepsilon') \quad 2.5.76;$$

once again, from our definition in 2.5.54 we obtain

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda', \varphi') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{t}'}, \ell_{\hat{\ell}'}) \quad 2.5.77;$$

on the other hand, since  $|T|/|\mathfrak{q}(n)| < |\mathfrak{t}(\ell)|/|\mathfrak{q}(n)|$ , it follows from the induction hypothesis applied to 2.5.76 that

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda', \varphi') = \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu', \varepsilon') \quad 2.5.78.$$

Consequently, we obtain

$$\begin{aligned} \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\lambda, \varphi) &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu', \varepsilon') \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mathrm{id}_{\mathfrak{t}'}, \ell_{\hat{\ell}'}) \\ &= \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\nu, \delta) \circ \widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{y}}}(\mu, \varepsilon) \end{aligned} \quad 2.5.79.$$

We are done.



2.6. As we mention in [5, 2.4], in order to see that a *folder structure* of  $\mathcal{F}$  induces a *folder structure* of  $N_{\mathcal{F}}(U)$  for any subgroup  $U$  of  $P$  fully normalized in  $\mathcal{F}$ , we have to consider the  $\mathcal{F}$ -*radical* subgroups of  $P$ . Recall that a subgroup  $R$  of  $P$  is  $\mathcal{F}$ -*radical* if it is  $\mathcal{F}$ -selfcentralizing and we have

$$\mathbf{O}_p(\tilde{\mathcal{F}}(R)) = \{1\} \quad 2.6.1$$

where we set  $\tilde{\mathcal{F}}(R) = \mathcal{F}(R)/\mathcal{F}_R(R)$  [7, 1.3]; let us denote by  $\mathcal{F}^{\text{rd}}$  the full subcategory of  $\mathcal{F}$  over the set of  $\mathcal{F}$ -radical subgroups of  $P$ , so that we have the inclusion of categories

$$\mathcal{F}^{\text{rd}} \subset \mathcal{F}^{\text{sc}} \subset \mathcal{F}^{\text{nc}} \quad 2.6.2.$$

2.7. Then, it follows from [5, Lemma 2.5] that, for any subgroup  $U$  of  $P$  fully normalized in  $\mathcal{F}$ , we get the inclusion  $N_{\mathcal{F}}(U)^{\text{rd}} \subset \mathcal{F}^{\text{sc}}$  and therefore a *folder structure*  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{sc}}}$  of  $\mathcal{F}$  induces by restriction a lifting

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{rd}}} : \mathbf{ch}^*(N_{\mathcal{F}}(U)^{\text{rd}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 2.7.1$$

of the canonical functor  $\mathbf{aut}_{N_{\mathcal{F}}(U)^{\text{rd}}} : \mathbf{ch}^*(N_{\mathcal{F}}(U)^{\text{rd}}) \rightarrow \mathfrak{Gr}$ ; moreover, denote by  $N_{\mathcal{F}^{\text{nc}}}(U)$  the intersection of  $N_{\mathcal{F}}(U)$  with  $\mathcal{F}^{\text{nc}}$  and by  $\mathbf{i}_U^{\text{nc}}$  the corresponding inclusion  $N_{\mathcal{F}^{\text{nc}}}(U) \subset \mathcal{F}^{\text{nc}}$ . At this point, it follows from [5, Theorem 2.9] and Theorem 2.5 above that we obtain a *folder structure* of  $N_{\mathcal{F}}(U)$ .

**Corollary 2.8.** *For any subgroup  $U$  of  $P$  fully normalized in  $\mathcal{F}$ , any folded Frobenius  $P$ -category  $(\mathcal{F}, \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}})$  induces a unique folded Frobenius  $N_P(U)$ -category  $(N_{\mathcal{F}}(U), \widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{nc}}})$  such that we have a natural  $k^*$ -map*

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)} \longrightarrow \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}} \circ \mathbf{ch}^*(\mathbf{i}_U^{\text{nc}}) \quad 2.8.1$$

lifting the canonical natural map  $\mathbf{aut}_{N_{\mathcal{F}^{\text{nc}}}(U)} \rightarrow \mathbf{aut}_{\mathcal{F}^{\text{nc}}} \circ \mathbf{ch}^*(\mathbf{i}_U^{\text{nc}})$ .

**Proof:** It follows from [5, Theorem 2.9] that the lifting  $\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{rd}}}$  in 2.7.1 above can be extended to a unique functor

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{sc}}} : \mathbf{ch}^*(N_{\mathcal{F}}(U)^{\text{sc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 2.8.2$$

lifting  $\mathbf{aut}_{N_{\mathcal{F}}(U)^{\text{sc}}}$ ; then, it follows from Theorem 2.5 above that, once again, the functor 2.8.2 can be extended to a unique functor

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{nc}}} : \mathbf{ch}^*(N_{\mathcal{F}}(U)^{\text{nc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 2.8.3$$

lifting  $\mathbf{aut}_{N_{\mathcal{F}}(U)^{\text{nc}}}$ ; in particular, by restriction we get a functor

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)} : \mathbf{ch}^*(N_{\mathcal{F}^{\text{nc}}}(U)) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 2.8.4.$$

But, we also can consider the *pull-back*

$$\begin{array}{ccc} \mathbf{aut}_{N_{\mathcal{F}^{\text{nc}}}(U)} & \longrightarrow & \mathbf{aut}_{\mathcal{F}^{\text{nc}}} \circ \mathbf{ch}^*(\mathbf{i}_U^{\text{nc}}) \\ \uparrow & & \uparrow \\ \widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)} & \longrightarrow & \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}} \circ \mathbf{ch}^*(\mathbf{i}_U^{\text{nc}}) \end{array} \quad 2.8.5$$

defining the functor  $\widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)}$  which still extends  $\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(U)^{\text{rd}}}$ ; then, it is not difficult to adapt the uniqueness arguments in the proofs of [5, Theorem 2.9] and Theorem 2.5 above to get a proof of the existence of a *natural  $k^*$ -isomorphism*

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)} \cong \widehat{\mathbf{aut}}_{N_{\mathcal{F}^{\text{nc}}}(U)} \quad 2.8.6;$$

now, the *natural  $k^*$ -map* 2.8.1 comes from this *natural  $k^*$ -isomorphism* and the bottom *natural  $k^*$ -map* in 2.8.5.

### 3. Regular central $k^*$ -extensions of $\mathcal{F}^{\text{sc}}$

3.1. An obvious way of getting a *folder structure* of  $\mathcal{F}$  is to start with a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}^{\text{nc}}$  of  $\mathcal{F}^{\text{nc}}$ ; indeed, in this case it follows from [4, Proposition A2.10] that we have a canonical functor

$$\mathbf{aut}_{\hat{\mathcal{F}}^{\text{sc}}} : \mathbf{ch}^*(\hat{\mathcal{F}}^{\text{nc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 3.1.1$$

mapping any  $\hat{\mathcal{F}}^{\text{nc}}$ -chain  $\hat{\mathbf{q}} : \Delta_n \rightarrow \hat{\mathcal{F}}^{\text{nc}}$  to the stabilizer  $\hat{\mathcal{F}}^{\text{nc}}(\mathbf{q})$  in  $\hat{\mathcal{F}}^{\text{nc}}(\mathbf{q}(n))$  of all the subgroups  $\text{Im}(\mathbf{q}(i \bullet n))$  for  $i \in \Delta_n$ , where  $\mathbf{q} : \Delta_n \rightarrow \mathcal{F}^{\text{nc}}$  denotes the corresponding  $\mathcal{F}^{\text{nc}}$ -chain; then, it is clear that this functor factorizes throughout a *folder structure* of  $\mathcal{F}$

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}} : \mathbf{ch}^*(\mathcal{F}^{\text{nc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 3.1.2.$$

Conversely, our main purpose is to prove that any *folder structure*  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}}$  of  $\mathcal{F}$  (cf. 2.4) comes from a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}^{\text{nc}}$  of  $\mathcal{F}^{\text{nc}}$ ; consequently, once this result will be obtained, to consider a *folded Frobenius  $P$ -category* is equivalent to consider a pair  $(\mathcal{F}, \hat{\mathcal{F}}^{\text{nc}})$  formed by a *Frobenius  $P$ -category*  $\mathcal{F}$  and by a *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}^{\text{nc}}$  of  $\mathcal{F}^{\text{nc}}$ .

3.2. Recall that if  $Q$  and  $Q'$  are  $\mathcal{F}$ -isomorphic  $\mathcal{F}$ -nilcentralized subgroups of  $P$ , for any pair of  $\mathcal{F}$ -nilcentralized subgroups  $R$  of  $Q$  and  $R'$  of  $Q'$  condition 2.1.1 in  $\mathcal{F}$  induces a *restriction map*

$$r_{R',R}^{Q',Q} : \mathcal{F}(Q', Q)_{R',R} \longrightarrow \mathcal{F}(R', R) \quad 3.2.1$$

where  $\mathcal{F}(Q', Q)_{R',R}$  denotes the set of  $\theta \in \mathcal{F}(Q', Q)$  such that  $\theta(R) \subset R'$ ; in particular, we have a group homomorphism from the stabilizer  $\mathcal{F}(Q)_R$  of  $R$  in  $\mathcal{F}(Q)$  to  $\mathcal{F}(R)$ . First of all, note the following consequence of condition 2.1.3.

**Lemma 3.3.** *With the notation above, assume that  $R$  and  $R'$  are  $\mathcal{F}$ -isomorphic and fully normalized in  $\mathcal{F}$ ; set  $N = N_P(R)$  and  $N' = N_P(R')$ . Then the restriction map and the composition induce a bijection*

$$\mathcal{F}(N', N)_{R',R} \times_{\mathcal{F}(N)_R} \mathcal{F}(R) \cong \mathcal{F}(R', R) \quad 3.3.1.$$

**Proof:** It is clear that if  $\theta \in \mathcal{F}(N', N)_{R', R}$  and  $\sigma \in \mathcal{F}(R)$  then the composition  $r_{R', R}^{N', N}(\theta) \circ \sigma$  belongs to  $\mathcal{F}(R', R)$ ; moreover, if  $\eta \in \mathcal{F}(N', N)_{R', R}$  and  $\tau \in \mathcal{F}(R)$  fulfill  $r_{R', R}^{N', N}(\eta) \circ \tau = r_{R', R}^{N', N}(\theta) \circ \sigma$ , then we have

$$r_{R, R}^{N, N}(\theta^{-1} \circ \eta) = \sigma \circ \tau^{-1} \quad 3.3.2$$

which implies that  $\theta^{-1} \circ \eta$  belongs to  $\mathcal{F}(N)_R$ ; consequently, the pairs  $(\theta, \sigma)$  and  $(\eta, \tau)$  have the same image in the quotient set

$$\mathcal{F}(N', N)_{R', R} \times_{\mathcal{F}(N)_R} \mathcal{F}(R) = (\mathcal{F}(N', N)_{R', R} \times \mathcal{F}(R)) / \mathcal{F}(N)_R \quad 3.3.3.$$

Conversely, any  $\theta \in \mathcal{F}(R', R)$  induces by conjugation a group isomorphism  $\mathcal{F}(R) \cong \mathcal{F}(R')$ ; then, since  $\mathcal{F}_N(R)$  and  $\mathcal{F}_{N'}(R')$  are respective Sylow  $p$ -subgroups of  $\mathcal{F}(R)$  and  $\mathcal{F}(R')$  [4, 2.11.4], there is  $\sigma \in \mathcal{F}(R)$  such that the isomorphism  $\mathcal{F}(R) \cong \mathcal{F}(R')$  induced by  $\theta \circ \sigma$  sends  $\mathcal{F}_N(R)$  onto  $\mathcal{F}_{N'}(R')$ ; at this point, it follows from condition 2.1.3 that there is  $\eta \in \mathcal{F}(N', N)$  such that  $r_{R', R}^{N', N}(\eta) = \theta \circ \sigma$ , so that  $\eta$  belongs to  $\mathcal{F}(N', N)_{R', R}$  and  $\theta$  is the image of the pair  $(\eta, \sigma^{-1})$ .

3.4. As in the proof of Theorem 2.5 above, in order to discuss the existence and the uniqueness of the announced  $k^*$ -category  $\hat{\mathcal{F}}^{\text{nc}}$ , we have to consider the situation relative to a nonempty set  $\mathfrak{X}$  of  $\mathcal{F}$ -nilcentralized subgroups of  $P$  which contains any subgroup of  $P$  admitting an  $\mathcal{F}$ -morphism from some subgroup in  $\mathfrak{X}$ ; more explicitly, we denote by  $\mathcal{F}^{\mathfrak{X}}$  the *full* subcategory of  $\mathcal{F}^{\text{nc}}$  over the set  $\mathfrak{X}$  of objects and consider the obvious restricted functor

$$\widehat{\text{aut}}_{\mathcal{F}^{\mathfrak{X}}} : \text{ch}^*(\mathcal{F}^{\mathfrak{X}}) \longrightarrow k^*\text{-Gr} \quad 3.4.1;$$

our main purpose is to prove that this functor comes from an essentially unique *regular central  $k^*$ -extension*  $\hat{\mathcal{F}}^{\mathfrak{X}}$  of  $\mathcal{F}^{\mathfrak{X}}$ ; actually, the uniqueness depends on the choice of a lifting  $i_Q^P \in \hat{\mathcal{F}}^{\mathfrak{X}}(P, Q)$  of the inclusion map  $\iota_Q^P : Q \rightarrow P$  when  $Q$  runs over  $\mathfrak{X}$ , as shows the next lemma.

**Lemma 3.5.** *With the notation above, any  $k^*$ -functor  $\hat{\mathfrak{f}}^{\mathfrak{X}} : \hat{\mathcal{F}}^{\mathfrak{X}} \rightarrow \hat{\mathcal{F}}^{\mathfrak{X}}$  lifting the identity on  $\mathcal{F}^{\mathfrak{X}}$ , fulfilling  $\hat{\mathfrak{f}}^{\mathfrak{X}}(i_Q^P) = i_Q^P$  for any  $Q \in \mathfrak{X}$  and such that the composition  $\widehat{\text{aut}}_{\hat{\mathcal{F}}^{\mathfrak{X}}} \circ \text{ch}^*(\hat{\mathfrak{f}}^{\mathfrak{X}})$  still lifts to  $\widehat{\text{aut}}_{\mathcal{F}^{\mathfrak{X}}}$  coincides with the identity.*

**Proof:** According to our hypothesis, for any  $Q \in \mathfrak{X}$ , the  $k^*$ -group homomorphism  $\hat{\mathfrak{f}}^{\mathfrak{X}}(Q) : \hat{\mathcal{F}}^{\mathfrak{X}}(Q) \rightarrow \hat{\mathcal{F}}^{\mathfrak{X}}(Q)$  is equal to the identity map, and it suffices to prove that, for any  $\hat{\varphi} \in \hat{\mathcal{F}}^{\mathfrak{X}}(P, Q)$ , we have  $\hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\varphi}) = \hat{\varphi}$ . Denoting by  $\varphi$  the image of  $\hat{\varphi}$  in  $\mathcal{F}^{\mathfrak{X}}(P, Q)$  and employing the terminology in [4, 5.15], we argue by induction on the *length*  $\ell(\varphi)$  of  $\varphi$ ; if  $\ell(\varphi) = 0$  we have  $\varphi = \sigma \circ \iota_Q^P$  for some  $\sigma \in \mathcal{F}(P)$  [4, Corollary 5.14] and therefore we get  $\hat{\varphi} = \hat{\sigma} \cdot i_Q^P$  for a suitable  $\hat{\sigma} \in \hat{\mathcal{F}}^{\mathfrak{X}}(P)$ , so that  $\hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\varphi}) = \hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\sigma}) \cdot \hat{\mathfrak{f}}^{\mathfrak{X}}(i_Q^P) = \hat{\sigma} \cdot i_Q^P = \hat{\varphi}$ .

Otherwise, we have [4, 5.15.1]

$$\varphi = \iota_R^P \circ \tau \circ \eta \quad \text{and} \quad \ell(\iota_R^P \circ \eta) = \ell(\varphi) - 1 \quad 3.5.1$$

for some  $R \in \mathfrak{X}$ , some  $\eta \in \mathcal{F}(R, Q)$  and some  $\tau \in \mathcal{F}(R)$ , and therefore we get  $\hat{\varphi} = \hat{\iota}_R^P \cdot \hat{\tau} \cdot \hat{\eta}$  for suitable  $\hat{\tau} \in \hat{\mathcal{F}}^{\mathfrak{X}}(R)$  and  $\hat{\eta} \in \hat{\mathcal{F}}^{\mathfrak{X}}(R, Q)$  respectively lifting  $\tau$  and  $\eta$ ; then, by the induction hypothesis, we obtain

$$\hat{\iota}_R^P \cdot \hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\eta}) = \hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\iota}_R^P \cdot \hat{\eta}) = \hat{\iota}_R^P \cdot \hat{\eta} \quad 3.5.2$$

which forces  $\hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\eta}) = \hat{\eta}$  and therefore we also obtain  $\hat{\mathfrak{f}}^{\mathfrak{X}}(\hat{\varphi}) = \hat{\varphi}$ .

3.6. Note that in a regular central  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{X}}$  of  $\mathcal{F}^{\mathfrak{X}}$  any morphism also is a *monomorphism* and therefore it follows from condition 2.1.1 that, with the notation in 3.2 above we also have a  $k^*$ -restriction map

$$\hat{r}_{R',R}^{Q',Q} : \hat{\mathcal{F}}^{\mathfrak{X}}(Q', Q)_{R',R} \longrightarrow \hat{\mathcal{F}}^{\mathfrak{X}}(R', R) \quad 3.6.1$$

where  $\hat{\mathcal{F}}^{\mathfrak{X}}(Q', Q)_{R',R}$  is the converse image of  $\mathcal{F}(Q', Q)_{R',R}$  in  $\hat{\mathcal{F}}^{\mathfrak{X}}(Q', Q)$ .

**Theorem 3.7.** *With the notation above, there exists an essentially unique regular central  $k^*$ -extension  $\hat{\mathcal{F}}^{\text{nc}}$  of  $\mathcal{F}^{\text{nc}}$  inducing the folded Frobenius  $P$ -category  $(\mathcal{F}, \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}})$ .*

**Proof:** We choose a set  $\mathfrak{X}$  as in 3.4 above and, arguing by induction on  $|\mathfrak{X}|$ , we will prove that there exists a regular central  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{X}}$  of  $\mathcal{F}^{\mathfrak{X}}$  inducing the obvious restricted functor (cf. 3.4.1)

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{X}}} : \mathbf{ch}^*(\mathcal{F}^{\mathfrak{X}}) \longrightarrow k^*\text{-}\mathbf{Gr} \quad 3.7.1$$

and that such a  $\hat{\mathcal{F}}^{\mathfrak{X}}$ , endowed with a choice of a family  $\{\hat{\iota}_Q^P\}_{Q \in \mathfrak{X}}$  of liftings  $\hat{\iota}_Q^P \in \hat{\mathcal{F}}^{\mathfrak{X}}(P, Q)$  of the inclusion maps  $\iota_Q^P : Q \rightarrow P$ , is unique up to a unique  $k^*$ -isomorphism.

If  $\mathfrak{X} = \{P\}$  then  $\mathcal{F}^{\mathfrak{X}}$  has just one object  $P$  and its automorphism group is  $\mathcal{F}(P)$ ; then, the *folder structure* maps the trivial  $\mathcal{F}^{\text{nc}}$ -chain  $\Delta_0 \rightarrow \mathcal{F}^{\text{nc}}$  sending 0 to  $P$  on a  $k^*$ -group  $\hat{\mathcal{F}}(P)$ ; that is to say, we get a  $k^*$ -category  $\hat{\mathcal{F}}^{\mathfrak{X}}$  with one object  $P$  and with the  $k^*$ -group automorphism  $\hat{\mathcal{F}}(P)$ , which clearly induces the corresponding functor 3.7.1 again; the uniqueness is clear.

Otherwise, choose a minimal element  $U$  in  $\mathfrak{X}$  *fully normalized* in  $\mathcal{F}$  and set

$$\mathfrak{Y} = \mathfrak{X} - \{\theta(U) \mid \theta \in \mathcal{F}(P, U)\} \quad 3.7.2;$$

then, according to our induction hypothesis, there exists a regular central  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{Y}}$  of  $\mathcal{F}^{\mathfrak{Y}}$  inducing the obvious restricted functor (cf. 3.4.1)

$$\widehat{\mathbf{aut}}_{\mathcal{F}^{\mathfrak{Y}}} : \mathbf{ch}^*(\mathcal{F}^{\mathfrak{Y}}) \longrightarrow k^*\text{-}\mathbf{Gr} \quad 3.7.3$$

and such a  $k^*$ -category  $\hat{\mathcal{F}}^{\mathfrak{y}}$ , endowed with a choice of a family  $\{\iota_Q^P\}_{Q \in \mathfrak{y}}$  of liftings  $\iota_Q^P \in \hat{\mathcal{F}}^{\mathfrak{y}}(P, Q)$  of the inclusion maps  $\iota_Q^P: Q \rightarrow P$ , is unique up to a unique  $k^*$ -isomorphism.

If  $N_{\mathcal{F}}(U) = \mathcal{F}$  [4, Proposition 2.16] then it follows from [4, Proposition 19.5] that  $\mathcal{F}$  coincides with the Frobenius  $P$ -category  $\mathcal{F}_L$  associated with the  $\mathcal{F}$ -localizer  $L$  of  $U$ ; moreover, since  $U$  is  $\mathcal{F}$ -nilcentralized, we have  $C_{\mathcal{F}}(U) = \mathcal{F}_{C_P(U)}$  (cf. 2.2); thus, setting  $C = C_L(U)$  and identifying  $P$  with its canonical image in  $L$  [4, Remark 18.7], any  $p'$ -element of  $N_C(Q)$  centralizes  $Q$  for any  $p$ -subgroup  $Q$  of  $C$ , so that  $N_C(Q)/C_C(Q)$  is a  $p$ -group, and therefore we have [4, 11]

$$C = \mathbb{O}_{p'}(C) \rtimes C_P(U) \quad 3.7.4;$$

then, since  $\mathbb{O}_{p'}(C)$  has a trivial image in  $\mathcal{F}(U)$ , it follows from [4, Theorem 18.6 and Remark 18.7] that  $\mathbb{O}_{p'}(C)$  is trivial; that is to say, the  $\mathcal{F}$ -self-centralizing subgroup  $\hat{U} = U \cdot C_P(U)$  of  $P$  (cf. 2.2) is normal in  $L$  and therefore we also have  $\mathcal{F} = N_{\mathcal{F}}(\hat{U})$ . In particular, for any pair of subgroups  $Q$  and  $R$  in  $\mathfrak{X}$ , the restriction induces a bijection

$$\mathcal{F}(Q \cdot \hat{U}, R \cdot \hat{U})_{Q, R} / C_{\mathcal{F}(R \cdot \hat{U})_R}(R) \cong \mathcal{F}(Q, R) \quad 3.7.5$$

and, denoting by  $\tilde{\mathcal{F}}$  the *exterior quotient* of  $\mathcal{F}$  [4, 1.3], an injective map [4, Corollary 4.9]

$$\tilde{\mathcal{F}}(Q \cdot \hat{U}, R \cdot \hat{U}) \longrightarrow \tilde{\mathcal{F}}(\hat{U}) \quad 3.7.6.$$

But, the *folder structure* maps the trivial  $\mathcal{F}^{\text{nc}}$ -chain  $\Delta_0 \rightarrow \mathcal{F}^{\text{nc}}$  sending 0 to  $\hat{U}$  on a  $k^*$ -group  $\hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U})$ ; moreover,  $\mathcal{F}_{\hat{U}}(\hat{U})$  admits a canonical lifting to  $\hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U})$ , still noted  $\mathcal{F}_{\hat{U}}(\hat{U})$ ; then, the quotient  $\hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U})/\mathcal{F}_{\hat{U}}(\hat{U})$  determines a central  $k^*$ -extension  $\hat{\hat{\mathcal{F}}}^{\mathfrak{x}}(\hat{U})$  of  $\tilde{\mathcal{F}}(\hat{U})$  and therefore the injection 3.7.6 determines a  $k^*$ -set  $\hat{\hat{\mathcal{F}}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})$ , so that we can consider the *pull-back*

$$\begin{array}{ccc} \mathcal{F}(Q \cdot \hat{U}, R \cdot \hat{U}) & \longrightarrow & \tilde{\mathcal{F}}(Q \cdot \hat{U}, R \cdot \hat{U}) \\ \uparrow & & \uparrow \\ \hat{\mathcal{F}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U}) & \longrightarrow & \hat{\hat{\mathcal{F}}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U}) \end{array} \quad 3.7.7$$

which defines the  $k^*$ -set  $\hat{\hat{\mathcal{F}}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})$ .

On the other hand, consider the  $\mathcal{F}^{\text{nc}}$ -chains  $\Delta_0 \rightarrow \mathcal{F}^{\text{nc}}$  and  $\Delta_1 \rightarrow \mathcal{F}^{\text{nc}}$  sending 0 to  $R \cdot \hat{U}$  or to  $\hat{U}$ , 1 to  $R \cdot \hat{U}$  and  $0 \bullet 1$  to the inclusion  $\hat{U} \rightarrow R \cdot \hat{U}$ ; up to a suitable identification, the *folder structure* provides a  $k^*$ -group homomorphism

$$\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U}) = \hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})_{\hat{U}} \longrightarrow \hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U}) \quad 3.7.8;$$

in particular,  $\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})$  acts on  $\hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U})$  via the composition on the right-hand, clearly stabilizing the  $k^*$ -subset  $\hat{\mathcal{F}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})$ ; now, it is easily checked that this  $k^*$ -group acts on the  $k^*$ -set  $\hat{\mathcal{F}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})$  and, in particular, the stabilizer  $\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})_R$  of  $R$  still acts on the  $k^*$ -subset  $\hat{\mathcal{F}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})_{Q,R}$  of this  $k^*$ -set.

Finally, since  $R$  is  $\mathcal{F}$ -nilcentralized, it follows from [4, Corollary 4.7] that  $C_{\mathcal{F}(R \cdot \hat{U})_R}(R)$  is a  $p$ -group and, as above, it has a canonical lifting to  $\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})_R$ , still noted  $C_{\mathcal{F}(R \cdot \hat{U})_R}(R)$ ; hence, according to bijection 3.7.5, we can define

$$\hat{\mathcal{F}}^{\mathfrak{x}}(Q, R) = \hat{\mathcal{F}}^{\mathfrak{x}}(Q \cdot \hat{U}, R \cdot \hat{U})_{Q,R} / C_{\mathcal{F}(R \cdot \hat{U})_R}(R) \quad 3.7.9.$$

Since the injective maps 3.7.6 are compatible with the composition in  $\tilde{\mathcal{F}}$ , it is routine to prove that definition 3.7.9 allows a lifting to the  $k^*$ -sets in 3.7.9 of this composition, defining a central regular  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{x}}$  of  $\mathcal{F}^{\mathfrak{x}}$ .

If  $\hat{\mathcal{F}}^{\mathfrak{x}}$  is another central regular  $k^*$ -extension of  $\mathcal{F}^{\mathfrak{x}}$  inducing the same *folder structure* and we have  $N_{\mathcal{F}}(U) = \mathcal{F}$ , note that for any  $Q \in \mathfrak{X}$  the  $p$ -group  $\mathcal{F}_Q(Q)$  has a canonical lifting to  $\hat{\mathcal{F}}^{\mathfrak{x}}(Q)$ , still noted  $\mathcal{F}_Q(Q)$ , and that then the  $k^*$ -sets defined by

$$\hat{\mathcal{F}}^{\mathfrak{x}}(Q, R) = \hat{\mathcal{F}}^{\mathfrak{x}}(Q, R) / \mathcal{F}_Q(Q) \quad 3.7.10,$$

where  $Q$  and  $R$  run over  $\mathfrak{X}$ , and the composition in  $\hat{\mathcal{F}}^{\mathfrak{x}}$  induce a central regular  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{x}}$  of  $\tilde{\mathcal{F}}^{\mathfrak{x}}$ ; moreover, following the steps above, it is clear that the *natural isomorphism* between the two *folder structures* provides a  $k^*$ -group isomorphism  $\hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U}) \cong \hat{\mathcal{F}}(\hat{U})$  and, for any  $R \in \mathfrak{X}$ , provides both a  $k^*$ -group isomorphism  $\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U}) \cong \hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})$  and a  $k^*$ -group homomorphism

$$\hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U}) = \hat{\mathcal{F}}^{\mathfrak{x}}(R \cdot \hat{U})_{\hat{U}} \longrightarrow \hat{\mathcal{F}}^{\mathfrak{x}}(\hat{U}) \quad 3.7.11$$

compatible with homomorphism 3.7.8. Now, for any  $Q$  and  $R$  in  $\mathfrak{X}$ , it is not difficult to get a  $k^*$ -set bijection

$$\hat{\mathcal{F}}^{\mathfrak{x}}(Q, R) \cong \hat{\mathcal{F}}^{\mathfrak{x}}(Q, R) \quad 3.7.12$$

in such a way that they are compatible with both compositions.

From now on, we assume that  $N_{\mathcal{F}}(U) \neq \mathcal{F}$ . It follows from Corollary 2.8 above that for any subgroup  $Q$  of  $P$  fully normalized in  $\mathcal{F}$ , our *folded Frobenius  $P$ -category*  $(\mathcal{F}, \widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}})$  induces a *folded Frobenius  $N_P(Q)$ -category*  $(N_{\mathcal{F}}(Q), \widehat{\mathbf{aut}}_{N_{\mathcal{F}}(Q)}^{\text{nc}})$  where

$$\widehat{\mathbf{aut}}_{N_{\mathcal{F}}(Q)}^{\text{nc}} : \mathbf{ch}^*(N_{\mathcal{F}}(Q)^{\text{nc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \quad 3.7.13$$

is the unique functor lifting  $\mathbf{aut}_{N_{\mathcal{F}}(Q)}^{\text{nc}}$  and extending the restriction of  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}}$  to  $N_{\mathcal{F}}(Q)^{\text{rd}}$ .

Thus, since  $N_{\mathcal{F}}(U) \neq \mathcal{F}$ , arguing by induction on the size of  $\mathcal{F}$ , for any  $V \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$  we may assume the existence of a unique *regular central  $k^*$ -extension*  $\widehat{N_{\mathcal{F}}(V)}^{\text{nc}}$  of  $N_{\mathcal{F}}(V)^{\text{nc}}$  determining  $\widehat{\text{aut}}_{N_{\mathcal{F}}(V)^{\text{nc}}}$ . Actually, denoting by  $N_{\mathfrak{X}}(V)$  the set of subgroups in  $\mathfrak{X}$  contained in  $N_P(V)$  (which are clearly  $N_{\mathcal{F}}(V)$ -nilcentralized!), we are only interested in the *full  $k^*$ -subcategory*  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathfrak{X}}(V)}$  of  $\widehat{N_{\mathcal{F}}(V)}^{\text{nc}}$  over  $N_{\mathfrak{X}}(V)$ .

Similarly, denoting by  $N_{\mathfrak{Y}}(V)$  the set of subgroups in  $\mathfrak{Y}$  contained in  $N_P(V)$ , we still consider the *full  $k^*$ -subcategory*  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathfrak{Y}}(V)}$  of  $\widehat{N_{\mathcal{F}}(V)}^{\text{nc}}$  over  $N_{\mathfrak{Y}}(V)$  which determines the *folder structure*  $\widehat{\text{aut}}_{N_{\mathcal{F}}(V)^{N_{\mathfrak{Y}}(V)}}$  of  $N_{\mathcal{F}}(V)^{N_{\mathfrak{Y}}(V)}$ ; but, it is easily checked from 3.7.3 that the  $k^*$ -subcategory  $N_{\hat{\mathcal{F}}^{\mathfrak{Y}}}(V)$  of  $\hat{\mathcal{F}}^{\mathfrak{Y}}$  still determines this *folder structure*; that is to say, once again according to our induction hypothesis, we may assume that the  $k^*$ -categories  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathfrak{Y}}(V)}$  and  $N_{\hat{\mathcal{F}}^{\mathfrak{Y}}}(V)$  coincide with each other, so that  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathfrak{Y}}(V)}$  is a  $k^*$ -subcategory of  $\hat{\mathcal{F}}^{\mathfrak{Y}}$ .

Moreover, setting  $N = N_P(V)$  and considering the  $N_{\mathcal{F}}(V)^{\text{nc}}$ -chains  $q_V: \Delta_0 \rightarrow N_{\mathcal{F}}(V)^{\text{nc}}$ ,  $q_N: \Delta_0 \rightarrow N_{\mathcal{F}}(V)^{\text{nc}}$  (cf. 2.3) and  $n: \Delta_1 \rightarrow N_{\mathcal{F}}(V)^{\text{nc}}$  which maps 0 on  $V$ , 1 on  $N$  and  $0 \bullet 1$  on  $\iota_V^N$ , together with the obvious  $\text{ch}^*(N_{\mathcal{F}}(V)^{\text{sc}})$ -morphisms (cf. 2.3)

$$(n, \Delta_1) \xrightarrow{(\text{id}_V, \delta_1^0)} (q_V, \Delta_0) \quad \text{and} \quad (n, \Delta_1) \xrightarrow{(\text{id}_N, \delta_0^0)} (q_N, \Delta_0) \quad 3.7.14,$$

it follows from 3.7.13 that the functors  $\widehat{\text{aut}}_{N_{\mathcal{F}}(V)^{\text{nc}}}$  and  $\widehat{\text{aut}}_{\mathcal{F}^{\text{nc}}}$  send  $(n, \Delta_1)$ ,  $(q_V, \Delta_0)$  and  $(q_N, \Delta_0)$  to the *same* respective  $k^*$ -groups, noted  $\widehat{\mathcal{F}(N)}_V$ ,  $\widehat{\mathcal{F}(V)}$  and  $\widehat{\mathcal{F}(N)}$ , and that they send the  $\text{ch}^*(N_{\mathcal{F}}(Q)^{\text{nc}})$ -morphisms  $(\text{id}, \delta_1^0)$  and  $(\text{id}_N, \delta_0^0)$  to the *same* respective  $k^*$ -group homomorphisms

$$\widehat{\mathcal{F}(N)}_V \longrightarrow \widehat{\mathcal{F}(V)} \quad \text{and} \quad \widehat{\mathcal{F}(N)}_V \longrightarrow \widehat{\mathcal{F}(N)} \quad 3.7.15;$$

note that the respective images of  $\widehat{\mathcal{F}(N)}_V$  are  $N_{\widehat{\mathcal{F}(V)}}(\mathcal{F}_N(V))$  and the stabilizer of  $V$  in  $\widehat{\mathcal{F}(N)}$ .

Since  $N$  belongs to  $\mathfrak{Y}$ ,  $\widehat{\mathcal{F}(N)}$  necessarily coincides with  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N)$  and therefore  $\widehat{\mathcal{F}(N)}_V$  also coincides with the stabilizer  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N)_V$  of  $V$  in  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N)$ . Then, for any  $V' \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$ , setting  $N' = N_P(V')$  and denoting by  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N', N)_{V', V}$  the converse image of  $\mathcal{F}(N', N)_{V', V}$  in  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N', N)$ , it is clear that  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N)_V$  acts on the  $k^*$ -set  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N', N)_{V', V}$  by right-hand composition in  $\hat{\mathcal{F}}^{\mathfrak{Y}}$ ; moreover, setting  $\hat{\mathcal{F}}^{\mathfrak{X}}(V) = \widehat{\mathcal{F}(V)}$ , the left-hand homomorphism in 3.7.15 induces a  $k^*$ -group homomorphism from  $\hat{\mathcal{F}}^{\mathfrak{Y}}(N)_V$  to  $\hat{\mathcal{F}}^{\mathfrak{X}}(V)$ ; thus,

we are able to define the  $k^*$ -set

$$\hat{\mathcal{F}}^{\mathfrak{x}}(V', V) = \hat{\mathcal{F}}^{\mathfrak{y}}(N', N)_{V', V} \times_{\hat{\mathcal{F}}^{\mathfrak{y}}(N)_V} \hat{\mathcal{F}}^{\mathfrak{x}}(V) \quad 3.7.16$$

and then, from isomorphism 3.3.1, we get a canonical map

$$\hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \longrightarrow \mathcal{F}(V', V) \quad 3.7.17.$$

Note that, in the case where  $V' = V$ , our notation is coherent and the pair formed by  $\hat{\iota}_N^N$  in  $\hat{\mathcal{F}}^{\mathfrak{y}}(N)_V$  and by the image of  $\hat{\iota}_N^N$  in  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$ , *via* the left-hand homomorphism in 3.7.15, coincides with the *unity* element  $\hat{\iota}_V^V$  in  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$ . Thus, for another  $V'' \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$ , setting  $N'' = N_P(V'')$  and considering  $\hat{\mathcal{F}}^{\mathfrak{y}}(N'', N)_{V'', V}$ ,  $\hat{\mathcal{F}}^{\mathfrak{y}}(N'', N')_{V'', V'}$  and  $\hat{\mathcal{F}}^{\mathfrak{x}}(V')$  as above, we also have the  $k^*$ -sets

$$\begin{aligned} \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V) &= \hat{\mathcal{F}}^{\mathfrak{y}}(N'', N)_{V'', V} \times_{\hat{\mathcal{F}}^{\mathfrak{y}}(N)_V} \hat{\mathcal{F}}^{\mathfrak{x}}(V) \\ \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') &= \hat{\mathcal{F}}^{\mathfrak{y}}(N'', N')_{V'', V'} \times_{\hat{\mathcal{F}}^{\mathfrak{y}}(N')_{V'}} \hat{\mathcal{F}}^{\mathfrak{x}}(V') \end{aligned} \quad 3.7.18$$

and we claim that the composition in  $\hat{\mathcal{F}}^{\mathfrak{y}}$  and in the corresponding  $k^*$ -groups induces a  $k^*$ -composition

$$\hat{\mathcal{F}}^{\mathfrak{x}}_{V'', V', V} : \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') \times \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \longrightarrow \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V) \quad 3.7.19$$

lifting the composition in  $\mathcal{F}$  *via* the canonical maps 3.7.17.

First of all, *mutatis mutandi* denote by  $\mathbf{q}_{V'}$ ,  $\mathbf{q}_{N'}$  and  $\mathbf{n}'$ , the analogous  $N_{\mathcal{F}}(V')^{\text{nc}}$ -chains and by  $(\mathbf{id}_{V'}, \delta_1^0)$  and  $(\mathbf{id}_{N'}, \delta_1^0)$  the analogous  $\mathbf{ch}^*(N_{\mathcal{F}}(V')^{\text{nc}})$ -morphisms; it is clear that any  $\mathcal{F}$ -morphism  $\varphi : N \rightarrow N'$  fulfilling  $\varphi(V) = V'$  determines *natural isomorphisms*  $\mathbf{q}_V \cong \mathbf{q}_{V'}$ ,  $\mathbf{q}_N \cong \mathbf{q}_{N'}$  and  $\mathbf{n} \cong \mathbf{n}'$  which induce commutative  $\mathbf{ch}^*(\mathcal{F}^{\text{nc}})$ -diagrams (cf. 3.7.14)

$$\begin{array}{ccc} (\mathbf{n}', \Delta_1) & \longrightarrow & (\mathbf{q}_{V'}, \Delta_0) \\ \wr \parallel & & \wr \parallel \\ (\mathbf{n}, \Delta_1) & \longrightarrow & (\mathbf{q}_V, \Delta_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathbf{n}', \Delta_1) & \longrightarrow & (\mathbf{q}_{N'}, \Delta_0) \\ \wr \parallel & & \wr \parallel \\ (\mathbf{n}, \Delta_1) & \longrightarrow & (\mathbf{q}_N, \Delta_0) \end{array} \quad 3.7.20;$$

at this point, the functor  $\widehat{\mathbf{aut}}_{\mathcal{F}^{\text{nc}}}$  sends these commutative  $\mathbf{ch}^*(\mathcal{F}^{\text{nc}})$ -diagrams to the commutative diagrams of  $k^*$ -groups

$$\begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{y}}(N')_{V'} & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{x}}(V') \\ \wr \parallel & \hat{\mathbf{g}}_{\varphi} \wr \parallel & \\ \hat{\mathcal{F}}^{\mathfrak{y}}(N)_V & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{x}}(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{y}}(N')_{V'} & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{y}}(N') \\ \wr \parallel & \hat{\mathbf{g}}_{\varphi} \wr \parallel & \\ \hat{\mathcal{F}}^{\mathfrak{y}}(N)_V & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{y}}(N) \end{array} \quad 3.7.21$$

and note that the  $k^*$ -group isomorphism  $\hat{\mathbf{g}}_{\varphi}$  has to be induced by the composition in the  $k^*$ -category  $\hat{\mathcal{F}}^{\mathfrak{y}}$  (cf. 3.7.3); that is to say, if  $\hat{\varphi} \in \hat{\mathcal{F}}^{\mathfrak{y}}(N', N)_{V', V}$  lifts  $\varphi$ , for any  $\hat{\sigma} \in \hat{\mathcal{F}}^{\mathfrak{y}}(N)$  we actually have  $\hat{\mathbf{g}}_{\varphi}(\hat{\sigma}) = \hat{\varphi} \cdot \hat{\sigma} \cdot \hat{\varphi}^{-1}$ .



We are ready to define the  $k^*$ -composition  $\hat{c}_{V'',V',V}^{\mathfrak{x}}$  in 3.7.19; any element in  $\hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  is the class  $\overline{(\hat{\varphi}, \hat{\sigma})}$  of some pair  $(\hat{\varphi}, \hat{\sigma})$  where  $\hat{\varphi}$  and  $\hat{\sigma}$  respectively belong to  $\hat{\mathcal{F}}^{\mathfrak{y}}(N', N)_{V', V}$  and to  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$ ; similarly, if  $\overline{(\hat{\varphi}', \hat{\sigma}')}$  is an element of  $\hat{\mathcal{F}}^{\mathfrak{x}}(V'', V')$  then it is clear that, in the  $k^*$ -category  $\hat{\mathcal{F}}^{\mathfrak{y}}$ , the composition  $\hat{\varphi}' \cdot \hat{\varphi}$  makes sense and belongs to  $\hat{\mathcal{F}}^{\mathfrak{y}}(N'', N)_{V'', V}$ ; moreover, denoting by  $\varphi$  the image of  $\hat{\varphi}$  in  $\mathcal{F}(N', N)$ , we have the  $k^*$ -group isomorphism  $\hat{h}_{\varphi}$  from  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$  to  $\hat{\mathcal{F}}^{\mathfrak{x}}(V')$  and therefore  $(\hat{h}_{\varphi})^{-1}(\hat{\sigma}')$  belongs to  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$ ; then, we set

$$\hat{c}_{V'',V',V}^{\mathfrak{x}}(\overline{(\hat{\varphi}', \hat{\sigma}')} , \overline{(\hat{\varphi}, \hat{\sigma})}) = \overline{(\hat{\varphi}' \cdot \hat{\varphi}, (\hat{h}_{\varphi})^{-1}(\hat{\sigma}') \cdot \hat{\sigma})} \quad 3.7.22;$$

the compatibility with the action of  $k^*$  is clear.

This makes sense since, for any  $\hat{\tau} \in \hat{\mathcal{F}}^{\mathfrak{y}}(N)_V$  and any  $\hat{\tau}' \in \hat{\mathcal{F}}^{\mathfrak{y}}(N')_{V'}$ , denoting by  $\tau$  the image of  $\hat{\tau}$  in  $\mathcal{F}(N)$ , in the category  $\hat{\mathcal{F}}^{\mathfrak{y}}$  and in the  $k^*$ -group  $\hat{\mathcal{F}}^{\mathfrak{x}}(V)$  we respectively get (cf. 3.7.21)

$$\begin{aligned} (\hat{\varphi}' \cdot \hat{\tau}') \cdot (\hat{\varphi} \cdot \hat{\tau}) &= \hat{\varphi}' \cdot \hat{\varphi} \cdot (\hat{g}_{\varphi})^{-1}(\hat{\tau}') \cdot \hat{\tau} \\ (\hat{h}_{\varphi \cdot \tau})^{-1}(\hat{\tau}'^{-1} \cdot \hat{\sigma}') \cdot (\hat{\tau}^{-1} \cdot \hat{\sigma}) &= ((\hat{h}_{\tau})^{-1} \circ (\hat{h}_{\varphi})^{-1})(\hat{\tau}'^{-1} \cdot \hat{\sigma}') \cdot (\hat{\tau}^{-1} \cdot \hat{\sigma}) \\ &= (\hat{h}_{\tau})^{-1}((\hat{g}_{\varphi})^{-1}(\hat{\tau}'^{-1}) \cdot (\hat{h}_{\varphi})^{-1}(\hat{\sigma}')) \cdot \hat{\tau}^{-1} \cdot \hat{\sigma} \\ &= \hat{\tau}^{-1} \cdot (\hat{g}_{\varphi})^{-1}(\hat{\tau}'^{-1}) \cdot (\hat{h}_{\varphi})^{-1}(\hat{\sigma}') \cdot \hat{\sigma} \\ &= ((\hat{g}_{\varphi})^{-1}(\hat{\tau}') \cdot \hat{\tau})^{-1} \cdot (\hat{h}_{\varphi})^{-1}(\hat{\sigma}') \cdot \hat{\sigma} \end{aligned} \quad 3.7.23;$$

consequently, we obtain

$$\hat{c}_{V'',V',V}^{\mathfrak{x}}(\overline{(\hat{\varphi}' \cdot \hat{\tau}', \hat{\tau}'^{-1} \cdot \hat{\sigma}')} , \overline{(\hat{\varphi} \cdot \hat{\tau}, \hat{\tau}^{-1} \cdot \hat{\sigma})}) = \hat{c}_{V'',V',V}^{\mathfrak{x}}(\overline{(\hat{\varphi}', \hat{\sigma}')} , \overline{(\hat{\varphi}, \hat{\sigma})}) \quad 3.7.24$$

This  $k^*$ -composition is associative since, for any  $V''' \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$  and any element  $\overline{(\hat{\varphi}'', \hat{\sigma}'')}$  in  $\hat{\mathcal{F}}^{\mathfrak{x}}(V''', V'')$ , denoting by  $\varphi'$  the image of  $\hat{\varphi}'$  in  $\mathcal{F}(N'', N')$  we obtain

$$\begin{aligned} \hat{c}_{V''',V'',V}^{\mathfrak{x}}\left(\overline{(\hat{\varphi}'', \hat{\sigma}'')} , \hat{c}_{V'',V',V}^{\mathfrak{x}}(\overline{(\hat{\varphi}', \hat{\sigma}')} , \overline{(\hat{\varphi}, \hat{\sigma})})\right) &= \hat{c}_{V''',V'',V}^{\mathfrak{x}}\left(\overline{(\hat{\varphi}'', \hat{\sigma}'')} , \overline{(\hat{\varphi}' \cdot \hat{\varphi}, (\hat{h}_{\varphi})^{-1}(\hat{\sigma}') \cdot \hat{\sigma})}\right) \\ &= \overline{(\hat{\varphi}'' \cdot (\hat{\varphi}' \cdot \hat{\varphi}), (\hat{h}_{\varphi' \cdot \varphi})^{-1}(\hat{\sigma}'') \cdot ((\hat{h}_{\varphi})^{-1}(\hat{\sigma}') \cdot \hat{\sigma}))} \\ &= \overline{((\hat{\varphi}'' \cdot \hat{\varphi}') \cdot \hat{\varphi}, (\hat{h}_{\varphi})^{-1}((\hat{h}_{\varphi'})^{-1}(\hat{\sigma}') \cdot \hat{\sigma}') \cdot \hat{\sigma})} \\ &= \hat{c}_{V''',V',V}^{\mathfrak{x}}\left(\hat{c}_{V'',V'',V'}^{\mathfrak{x}}(\overline{(\hat{\varphi}'', \hat{\sigma}'')} , \overline{(\hat{\varphi}', \hat{\sigma}')}), \overline{(\hat{\varphi}, \hat{\sigma})}\right) \end{aligned} \quad 3.7.25.$$

According to our definition of  $\hat{\mathcal{F}}^x(V', V)$  in 3.7.16, the unity element of  $\hat{\mathcal{F}}^x(V)$  defines a canonical  $k^*$ -set homomorphism

$$\hat{r}_{V',V}^{N',N} : \hat{\mathcal{F}}^y(N', N)_{V',V} \longrightarrow \hat{\mathcal{F}}^x(V', V) \quad 3.7.26$$

lifting  $r_{V',V}^{N',N}$ . More generally, let  $Q$  and  $Q'$  be a pair of subgroups of  $P$  respectively contained in  $N$  and  $N'$ , and strictly containing  $V$  and  $V'$ ; we define a  $k^*$ -set homomorphism

$$\hat{r}_{V',V}^{Q',Q} : \hat{\mathcal{F}}^y(Q', Q)_{V',V} \longrightarrow \hat{\mathcal{F}}^x(V', V) \quad 3.7.27$$

lifting the restriction map (cf. 3.2.1)

$$r_{V',V}^{Q',Q} : \mathcal{F}(Q', Q)_{V',V} \longrightarrow \mathcal{F}(V', V) \quad 3.7.28$$

as follows. If  $\hat{\varphi} \in \hat{\mathcal{F}}^y(Q', Q)_{V',V}$  and  $\varphi$  denotes its image in  $\mathcal{F}(Q', Q)_{V',V}$ , it follows from Lemma 3.3 that  $r_{V',V}^{Q',Q}(\varphi) = r_{V',V}^{N',N}(\psi) \circ \theta$  for suitable  $\psi$  in  $\mathcal{F}(N', N)_{V',V}$  and  $\theta$  in  $\mathcal{F}(V)$ ; thus, setting  $Q'' = \psi^{-1}(Q') \subset N$ , we get

$$\theta = r_{V,V}^{Q'',Q}(r_{Q'',Q'}^{N,N'}(\psi^{-1}) \circ \varphi) \quad 3.7.29$$

and therefore, setting  $\sigma = r_{Q'',Q'}^{N,N'}(\psi^{-1}) \circ \varphi$ , we still get  $\varphi = r_{Q',Q''}^{N',N}(\psi) \circ \sigma$ .

Hence, choosing a lifting  $\hat{\psi}$  of  $\psi$  in  $\hat{\mathcal{F}}^y(N', N)_{V',V}$ , in the  $k^*$ -category  $\hat{\mathcal{F}}^y$  we have the restriction  $\hat{r}_{Q',Q''}^{N',N}(\hat{\psi})$  (cf. 3.6) as an element of  $\hat{\mathcal{F}}^y(Q', Q'')_{V',V}$ ; then, in this  $k^*$ -category there is a unique lifting  $\hat{\sigma}$  to  $\hat{\mathcal{F}}^y(Q'', Q)_{V,V}$  of  $\sigma$  fulfilling

$$\hat{\varphi} = \hat{r}_{Q',Q''}^{N',N}(\hat{\psi}) \cdot \hat{\sigma} \quad 3.7.30.$$

Moreover, since  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathcal{F}}(V)}$  has been identified with the  $k^*$ -subcategory  $N_{\hat{\mathcal{F}}^y}(V)$  of  $\hat{\mathcal{F}}^y$  and since we have  $Q'' \subset N$  and  $\hat{\sigma} \in \hat{\mathcal{F}}^y(Q'', Q)_{V,V}$ , actually  $\hat{\sigma}$  can be identified with an element of  $\widehat{N_{\mathcal{F}}(V)}^{nc}(Q'', Q)$  and therefore in the  $k^*$ -category  $\widehat{N_{\mathcal{P}}(V)}^{N_{\mathcal{X}}(V)}$  we have the restriction  $\hat{r}_{V,V}^{Q'',Q}(\hat{\sigma})$  (cf. 3.6) lifting  $\theta$  to  $\widehat{N_{\mathcal{P}}(V)}^{N_{\mathcal{X}}(V)}(V)$  which coincides with  $\hat{\mathcal{F}}^x(V)$  since we have (cf. 3.7.14)

$$\widehat{N_{\mathcal{P}}(V)}^{nc}(V) = \widehat{\text{aut}}_{N_{\mathcal{F}}(V)}^{nc}(\mathbf{q}_V) = \widehat{\text{aut}}_{\mathcal{F}^{nc}}(\mathbf{q}_V) = \hat{\mathcal{F}}^x(V) \quad 3.7.31.$$

Then, we define (cf. 3.7.12)

$$\hat{r}_{V',V}^{Q',Q}(\hat{\varphi}) = \overline{(\hat{\psi}, \hat{r}_{V,V}^{Q'',Q}(\hat{\sigma}))} \quad 3.7.32.$$

This definition is independent of our choice of  $\psi \in \mathcal{F}(N', N)_{V', V}$  since, for another decomposition  $r_{V', V}^{Q', Q}(\varphi) = r_{V', V}^{N', N}(\psi') \circ \theta'$ , we actually obtain  $\psi' = \psi \circ \tau$  and  $\theta' = r_V^N(\tau^{-1}) \circ \theta$  for some  $\tau \in \mathcal{F}(N)_V$ ; consequently, setting  $Q''' = (\pi_N(\tau^{-1}))(Q'')$ , once again an element  $\hat{\tau}$  of  $\hat{\mathcal{F}}^{\mathfrak{y}}(N)_V$  lifting  $\tau$  can be identified with an element of  $\widehat{N_{\mathcal{F}}(V)}^{\text{nc}}(N)$  and we also obtain

$$\hat{\varphi} = \hat{r}_{Q', Q''}^{N', N}(\hat{\psi}) \cdot \hat{\sigma} = (\hat{r}_{Q', Q'''}^{N', N}(\hat{\psi} \cdot \hat{\tau})) \cdot (\hat{r}_{Q''', Q''}^{N, N}(\hat{\tau}^{-1}) \cdot \hat{\sigma}) \quad 3.7.33;$$

but, the pairs  $(\hat{\psi}, \hat{r}_{V, V}^{Q'', Q}(\hat{\sigma}))$  and  $(\hat{\varphi} \cdot \hat{\tau}, \hat{r}_{V, V}^{Q''', Q}(\hat{r}_{Q''', Q''}^{N, N}(\hat{\tau}^{-1}) \cdot \hat{\sigma}))$  actually have the same class in  $\hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$ .

At present, if  $R$  and  $R'$  are a pair of subgroups of  $P$  respectively contained in  $Q$  and  $Q'$ , and strictly containing  $V$  and  $V'$ , we claim that the corresponding restriction  $\hat{r}_{V', V}^{R', R}$  agree with  $\hat{r}_{V', V}^{Q', Q}$ ; that is to say, if  $\hat{\varphi}$  in  $\hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V}$  has an image in  $\mathcal{F}(Q', Q)$  mapping  $R$  on  $R'$ , it follows from 3.6 above that we have the restriction  $\hat{r}_{R', R}^{Q', Q}(\hat{\varphi})$  in  $\hat{\mathcal{F}}^{\mathfrak{y}}(R', R)_{V', V}$  and we claim that

$$\hat{r}_{V', V}^{R', R}(\hat{r}_{R', R}^{Q', Q}(\hat{\varphi})) = \hat{r}_{V', V}^{Q', Q}(\hat{\varphi}) \quad 3.7.34.$$

Indeed, with the notation above we may assume that  $\hat{\varphi} = \hat{r}_{Q', Q''}^{N', N}(\hat{\psi}) \cdot \hat{\sigma}$ ; then, setting  $R'' = \psi^{-1}(R') \subset N$ , we clearly have

$$\hat{r}_{R', R}^{Q', Q}(\hat{\varphi}) = \hat{r}_{R', R''}^{N', N}(\hat{\psi}) \cdot \hat{r}_{R'', R}^{Q'', Q}(\hat{\sigma}) \quad 3.7.35;$$

consequently, since the restriction in the  $k^*$ -category  $\widehat{N_{\mathcal{F}}(V)}^{N_{\mathfrak{x}}(V)}$  is transitive (cf. 3.6), we clearly obtain

$$\begin{aligned} \hat{r}_{V', V}^{R', R}(\hat{r}_{R', R}^{Q', Q}(\hat{\varphi})) &= \overline{(\hat{\psi}, \hat{r}_{V, V}^{R'', R}(\hat{r}_{R'', R}^{Q'', Q}(\hat{\sigma})))} = \overline{(\hat{\psi}, \hat{r}_{V, V}^{Q'', Q}(\hat{\sigma}))} \\ &= \hat{r}_{V', V}^{Q', Q}(\hat{\varphi}) \end{aligned} \quad 3.7.36.$$

As above, consider a third  $V'' \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$ , and a subgroup  $Q''$  of  $P$  contained in  $N'' = N_P(V'')$  and strictly containing  $V''$ ; thus, we have the three  $k^*$ -set homomorphisms  $\hat{r}_{V', V}^{Q', Q}$ ,  $\hat{r}_{V'', V'}^{Q'', Q'}$  and  $\hat{r}_{V'', V}^{Q'', Q}$  and we claim that they are compatible with the  $k^*$ -compositions (cf. 3.7.19), namely that we have the following commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{y}}(Q'', Q')_{V'', V'} \times \hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V} & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{y}}(Q'', Q)_{V'', V} \\ \hat{r}_{V'', V'}^{Q'', Q'} \times \hat{r}_{V', V}^{Q', Q} \downarrow & & \downarrow \hat{r}_{V'', V}^{Q'', Q} \\ \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') \times \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V) \end{array} \quad 3.7.37.$$

Indeed, let  $\hat{\varphi}$  and  $\hat{\varphi}'$  be respective elements of  $\hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V}$  and of  $\hat{\mathcal{F}}^{\mathfrak{y}}(Q'', Q')_{V'', V'}$ ; we actually may assume that

$$\hat{\varphi} = \hat{r}_{Q', R}^{N', N}(\hat{\psi}) \cdot \hat{\sigma} \quad \text{and} \quad \hat{\varphi}' = \hat{r}_{Q'', R'}^{N'', N'}(\hat{\psi}') \cdot \hat{\sigma}' \quad 3.7.38$$

where  $\hat{\psi}$  and  $\hat{\psi}'$  are suitable elements respectively belonging to  $\hat{\mathcal{F}}^{\mathfrak{y}}(N', N)_{V', V}$  and to  $\hat{\mathcal{F}}^{\mathfrak{y}}(N'', N')_{V'', V'}$ , and, denoting by  $\psi$  and  $\psi'$  their images in  $\mathcal{F}$  and setting

$$R = \psi^{-1}(Q') \quad \text{and} \quad R' = \psi'^{-1}(Q'') \quad 3.7.39,$$

where  $\hat{\sigma}$  and  $\hat{\sigma}'$  are suitable elements respectively belonging to  $\hat{\mathcal{F}}^{\mathfrak{y}}(R, Q)_{V', V}$  and to  $\hat{\mathcal{F}}^{\mathfrak{y}}(R', Q')_{V'', V'}$ . Then, setting  $R'' = \psi^{-1}(R') = (\psi' \circ \psi)^{-1}(Q'')$ , we clearly have

$$\begin{aligned} \hat{\varphi}' \cdot \hat{\varphi} &= (\hat{r}_{Q'', R'}^{N'', N'}(\hat{\psi}') \cdot \hat{\sigma}') \cdot (\hat{r}_{Q', R}^{N', N}(\hat{\psi}) \cdot \hat{\sigma}) \\ &= \hat{r}_{Q'', R'}^{N'', N'}(\hat{\psi}' \cdot \hat{\psi}) \cdot (\hat{r}_{R'', R'}^{N, N'}(\hat{\psi}^{-1}) \cdot \hat{\sigma}' \cdot \hat{r}_{Q', R}^{N', N}(\hat{\psi})) \cdot \hat{\sigma} \end{aligned} \quad 3.7.40.$$

Hence, setting  $\hat{\sigma}'' = \hat{r}_{R'', R'}^{N, N'}(\hat{\psi}^{-1}) \cdot \hat{\sigma}' \cdot \hat{r}_{Q', R}^{N', N}(\hat{\psi})$ , we get (cf. 3.7.27)

$$\hat{r}_{V'', V}^{Q'', Q}(\hat{\varphi}' \cdot \hat{\varphi}) = \overline{(\hat{\psi}' \cdot \hat{\psi}, \hat{r}_{V, V}^{R'', Q}(\hat{\sigma}'' \cdot \hat{\sigma}))} \quad 3.7.41.$$

On the other hand, from equalities 3.7.33 we obtain (cf. 3.7.27)

$$\hat{r}_{V', V}^{Q', Q}(\hat{\varphi}) = \overline{(\hat{\psi}, \hat{r}_{V, V}^{R, Q}(\hat{\sigma}))} \quad \text{and} \quad \hat{r}_{V'', V'}^{Q'', Q'}(\hat{\varphi}') = \overline{(\hat{\psi}', \hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}'))} \quad 3.7.42;$$

but, according to our definition in 3.7.19, we get

$$\begin{aligned} \hat{c}_{V'', V', V}^{\mathfrak{x}} \left( \overline{(\hat{\psi}', \hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}'))}, \overline{(\hat{\psi}, \hat{r}_{V, V}^{R, Q}(\hat{\sigma}))} \right) \\ = \overline{(\hat{\psi}' \cdot \hat{\psi}, (\hat{\mathbf{h}}_{\psi})^{-1}(\hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}')) \cdot \hat{r}_{V, V}^{R, Q}(\hat{\sigma}))} \end{aligned} \quad 3.7.43$$

and then we claim that we have  $(\hat{\mathbf{h}}_{\psi})^{-1}(\hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}')) = \hat{r}_{V, V}^{R'', R}(\hat{\sigma}'')$  which forces

$$\begin{aligned} \hat{c}_{V'', V', V}^{\mathfrak{x}} \left( \overline{(\hat{\psi}', \hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}'))}, \overline{(\hat{\psi}, \hat{r}_{V, V}^{R, Q}(\hat{\sigma}))} \right) \\ = \overline{(\hat{\psi}' \cdot \hat{\psi}, \hat{r}_{V, V}^{R'', Q}(\hat{\sigma}'' \cdot \hat{\sigma}))} = \hat{r}_{V'', V}^{Q'', Q}(\hat{\varphi}' \cdot \hat{\varphi}) \end{aligned} \quad 3.7.44.$$

Indeed, denoting by  $\xi'$  the image of  $\hat{\xi}' = \hat{r}_{R'}^{N'} \cdot \hat{\sigma}'$  in  $(N_{\mathcal{F}}(V'))(N', Q')$  (cf. 3.7.1) and employing the terminology in [4, 5.15], we argue by induction on the *length*  $\ell(\xi')$  of  $\xi'$ ; if  $\ell(\xi') = 0$  we have  $\xi' = \nu' \circ \iota_{Q'}^{N'}$  for  $\nu'$  in  $(N_{\mathcal{F}}(V'))(N')$  [4, Corollary 5.14] and therefore we get  $\hat{\xi}' = \hat{\nu}' \cdot \hat{r}_{Q'}^{N'}$  for a suitable  $\hat{\nu}'$  in  $\hat{\mathcal{F}}^{\mathfrak{y}}(N')_{V'}$ , so that we obtain (cf. 3.7.21)

$$(\hat{\mathbf{h}}_{\psi})^{-1}(\hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}')) = \hat{r}_{V'}^{N'}(\hat{\mathbf{g}}_{\psi}(\hat{\nu}')) = \hat{r}_{V'}^{N'}(\hat{\psi} \cdot \hat{\nu}' \cdot \hat{\psi}^{-1}) \quad 3.7.45.$$

Otherwise, we have [4, 5.15.1]

$$\xi' = \iota_{T'}^{N'} \circ \mu' \circ \eta' \quad \text{and} \quad \ell(\iota_{T'}^{N'} \circ \eta') = \ell(\xi') - 1 \quad 3.7.46$$

for suitable  $T' \in N_{\mathfrak{Y}}(V')$ ,  $\eta' \in (N_{\mathcal{F}}(V'))(T', Q')$  and  $\mu' \in (N_{\mathcal{F}}(V'))(T')$ , and therefore we get  $\hat{\xi}' = \hat{\iota}_{T'}^{N'} \cdot \hat{\mu}' \cdot \hat{\eta}'$  for suitable elements  $\hat{\mu}' \in \hat{\mathcal{F}}^{\mathfrak{Y}}(T')_{V'}$  and  $\hat{\eta}' \in \hat{\mathcal{F}}^{\mathfrak{Y}}(T', Q')_{V', V'}$  respectively lifting  $\mu'$  and  $\eta'$ ; hence, we obtain

$$\hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}') = \hat{r}_{V'}^{T'}(\hat{\mu}') \cdot \hat{r}_{V', V'}^{T', Q'}(\hat{\eta}') \quad 3.7.47$$

and therefore we still obtain

$$(\hat{h}_{\psi})^{-1}(\hat{r}_{V', V'}^{R', Q'}(\hat{\sigma}')) = (\hat{h}_{\psi})^{-1}(\hat{r}_{V'}^{T'}(\hat{\mu}')) \cdot (\hat{h}_{\psi})^{-1}(\hat{r}_{V', V'}^{T', Q'}(\hat{\eta}')) \quad 3.7.48.$$

Then, by the induction hypothesis, setting

$$T = \psi^{-1}(T') \quad \text{and} \quad \hat{\eta}'' = \hat{r}_{T, T'}^{N, N'}(\hat{\psi}^{-1}) \cdot \hat{\eta}' \cdot \hat{r}_{Q', R}^{N', N}(\hat{\psi}) \quad 3.7.49,$$

we have  $(\hat{h}_{\psi})^{-1}(\hat{r}_{V', V'}^{T', Q'}(\hat{\eta}')) = \hat{r}_{V, V}^{T, R}(\hat{\eta}'')$ ; moreover, it is quite clear that replacing  $N$  by  $T$  and  $N'$  by  $T'$  in 3.7.21, we still get the commutative diagrams of  $k^*$ -groups

$$\begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{Y}}(T')_{V'} & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{X}}(V') \\ \wr \parallel & & \wr \parallel \\ \hat{\mathcal{F}}^{\mathfrak{Y}}(T)_V & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{X}}(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{Y}}(T')_{V'} & \longrightarrow & \hat{\mathcal{P}}^{\mathfrak{Y}}(T') \\ \wr \parallel & & \wr \parallel \\ \hat{\mathcal{F}}^{\mathfrak{Y}}(T)_V & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{Y}}(T) \end{array} \quad 3.7.50$$

and thus, since  $\hat{\mu}'$  belongs to  $\hat{\mathcal{F}}^{\mathfrak{Y}}(T')_{V'}$ , setting

$$\hat{\mu}'' = \hat{r}_{T, T'}^{N, N'}(\hat{\psi}^{-1}) \cdot \hat{\mu}' \cdot \hat{r}_{T', T}^{N', N}(\hat{\psi}) \quad 3.7.51,$$

we still have  $(\hat{h}_{\psi})^{-1}(\hat{r}_{V', V'}^{T', Q'}(\hat{\mu}')) = \hat{r}_V^T(\hat{\mu}'')$ . Finally, it is easily checked that

$$\hat{r}_{V, V}^{R', R}(\hat{\sigma}'') = \hat{r}_V^T(\hat{\mu}'') \cdot \hat{r}_{V, V}^{T, R}(\hat{\eta}'') \quad 3.7.52,$$

which completes the proof of our claim.

We are ready to define the  $k^*$ -set  $\hat{\mathcal{F}}^{\mathfrak{X}}(V', V)$  for any pair of subgroups  $V$  and  $V'$  in  $\mathfrak{X} - \mathfrak{Y}$ ; we clearly have  $N = N_P(V) \neq V$  and it follows from [4, Proposition 2.7] that there is an  $\mathcal{F}$ -morphism  $\nu : N \rightarrow P$  such that  $\nu(V)$  is fully normalized in  $\mathcal{F}$ ; moreover, we choose  $\hat{\nu} \in \hat{\mathcal{F}}^{\mathfrak{Y}}(\nu(N), N)$  lifting the  $\mathcal{F}$ -isomorphism  $\nu_*$  determined by  $\nu$ . That is to say, we may assume that

**3.7.53** *There is a pair  $(N, \hat{\nu})$  formed by a subgroup  $N$  of  $P$  which strictly contains and normalizes  $V$ , and by an element  $\hat{\nu}$  in  $\hat{\mathcal{F}}^{\mathfrak{Y}}(\nu(N), N)$  lifting the  $\mathcal{F}$ -isomorphism  $\nu_* : N \cong \nu(N)$  determined by a  $\mathcal{F}$ -morphism  $\nu : N \rightarrow P$  such that  $\nu(V)$  is fully normalized in  $\mathcal{F}$ .*

We denote by  $\hat{\mathfrak{N}}(V)$  the set of such pairs and often we write  $\hat{\nu}$  instead of  $(N, \hat{\nu})$ , setting  ${}^{\nu}N = \nu(N)$  and  ${}^{\nu}V = \nu(V)$ .

For another pair  $(\bar{N}, \hat{\nu})$  in  $\hat{\mathfrak{N}}(V)$ , denoting by  $\bar{\nu}: \bar{N} \rightarrow P$  the  $\mathcal{F}$ -morphism determined by  $\hat{\nu}$ , setting  $M = \langle N, \bar{N} \rangle$  and considering a new  $\mathcal{F}$ -morphism  $\mu: M \rightarrow P$  such that  $\mu(V)$  is fully normalized in  $\mathcal{F}$ , we can obtain a third pair  $(M, \hat{\mu})$  in  $\hat{\mathfrak{N}}(V)$ ; then,  $\hat{r}_{\mu N, N}^{\mu M, M}(\hat{\mu}) \cdot \hat{\nu}^{-1}$  and  $\hat{r}_{\mu \bar{N}, \bar{N}}^{\mu M, M}(\hat{\mu}) \cdot \hat{\bar{\nu}}^{-1}$  respectively belong to  $\hat{\mathcal{F}}^{\mathfrak{y}}(\mu N, {}^{\nu}N)$  and to  $\hat{\mathcal{F}}^{\mathfrak{y}}(\mu \bar{N}, {}^{\bar{\nu}}\bar{N})$ ; in particular, since  ${}^{\nu}V, {}^{\bar{\nu}}V$  and  ${}^{\mu}V$  are fully normalized in  $\mathcal{F}$ , the  $k^*$ -sets  $\hat{\mathcal{F}}^{\mathfrak{x}}(\mu V, {}^{\nu}V)$ ,  $\hat{\mathcal{F}}^{\mathfrak{x}}(\mu V, {}^{\bar{\nu}}V)$  and  $\hat{\mathcal{F}}^{\mathfrak{x}}({}^{\bar{\nu}}V, {}^{\nu}V)$  have been already defined above, and we consider the element

$$\hat{g}_{\hat{\bar{\nu}}, \hat{\nu}} = \hat{r}_{\mu V, {}^{\bar{\nu}}V}^{\mu \bar{N}, {}^{\bar{\nu}}\bar{N}}(\hat{r}_{\mu \bar{N}, \bar{N}}^{\mu M, M}(\hat{\mu}) \cdot \hat{\bar{\nu}}^{-1})^{-1} \cdot \hat{r}_{\mu V, {}^{\nu}V}^{\mu N, {}^{\nu}N}(\hat{r}_{\mu N, N}^{\mu M, M}(\hat{\mu}) \cdot \hat{\nu}^{-1}) \quad 3.7.54$$

in  $\hat{\mathcal{F}}^{\mathfrak{x}}({}^{\bar{\nu}}V, {}^{\nu}V)$ , which actually does not depend on the choice of  $\mu$ .

Indeed, for another pair  $(M, \hat{\mu}')$  in  $\mathfrak{N}(V)$  we have

$$\begin{aligned} \hat{r}_{\mu' N, N}^{\mu' M, M}(\hat{\mu}') &= \hat{r}_{\mu' N, {}^m N}^{\mu' M, {}^m M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu N, N}^{{}^m M, M}(\hat{\mu}) \\ \hat{r}_{\mu' \bar{N}, \bar{N}}^{\mu' M, M}(\hat{\mu}') &= \hat{r}_{\mu' \bar{N}, {}^{\mu} \bar{N}}^{\mu' M, {}^{\mu} M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu \bar{N}, \bar{N}}^{\mu M, M}(\hat{\mu}) \end{aligned} \quad 3.7.55$$

and therefore it follows from equality 3.7.34 that we get

$$\begin{aligned} &\hat{r}_{\mu' V, {}^{\nu}V}^{\mu' N, {}^{\nu}N}(\hat{r}_{\mu' N, N}^{\mu' M, M}(\hat{\mu}') \cdot \hat{\nu}^{-1}) \\ &= \hat{r}_{\mu' V, {}^{\nu}V}^{\mu' N, {}^{\nu}N}(\hat{r}_{\mu' N, {}^{\mu}N}^{\mu' M, {}^{\mu}M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu N, N}^{\mu M, M}(\hat{\mu}) \cdot \hat{\nu}^{-1}) \\ &= \hat{r}_{\mu' V, {}^{\mu}V}^{\mu' M, {}^{\mu}M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu' V, {}^{\nu}V}^{\mu' N, {}^{\nu}N}(\hat{r}_{\mu N, N}^{\mu M, M}(\hat{\mu}) \cdot \hat{\nu}^{-1}) \\ &\hat{r}_{\mu' V, {}^{\bar{\nu}}V}^{\mu' \bar{N}, {}^{\bar{\nu}}\bar{N}}(\hat{r}_{\mu' \bar{N}, \bar{N}}^{\mu' M, M}(\hat{\mu}') \cdot \hat{\bar{\nu}}^{-1}) \\ &= \hat{r}_{\mu' V, {}^{\bar{\nu}}V}^{\mu' \bar{N}, {}^{\bar{\nu}}\bar{N}}(\hat{r}_{\mu' \bar{N}, {}^{\mu}\bar{N}}^{\mu' M, {}^{\mu}M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu \bar{N}, \bar{N}}^{\mu M, M}(\hat{\mu}) \cdot \hat{\bar{\nu}}^{-1}) \\ &= \hat{r}_{\mu' V, {}^{\mu}V}^{\mu' M, {}^{\mu}M}(\hat{\mu}' \cdot \hat{\mu}^{-1}) \cdot \hat{r}_{\mu' V, {}^{\bar{\nu}}V}^{\mu' \bar{N}, {}^{\bar{\nu}}\bar{N}}(\hat{r}_{\mu \bar{N}, \bar{N}}^{\mu M, M}(\hat{\mu}) \cdot \hat{\bar{\nu}}^{-1}) \end{aligned} \quad 3.7.56,$$

which proves our claim. Similarly, for any triple of pairs  $(N, \hat{\nu})$ ,  $(\bar{N}, \hat{\bar{\nu}})$  and  $(\langle N, \bar{N}, \bar{\bar{N}} \rangle, \hat{\mu})$  in  $\hat{\mathfrak{N}}(V)$ , considering a pair  $(\langle N, \bar{N}, \bar{\bar{N}} \rangle, \hat{\mu})$  in  $\hat{\mathfrak{N}}(V)$ , it follows from equality 3.7.34 and from the commutativity of diagram 3.7.37 that

$$\hat{g}_{\hat{\bar{\nu}}, \hat{\nu}} \cdot \hat{g}_{\hat{\nu}, \hat{\bar{\nu}}} = \hat{g}_{\hat{\bar{\nu}}, \hat{\bar{\nu}}} \quad 3.7.57.$$

Note that if  $V$  is fully normalized in  $\mathcal{F}$  then the pair formed by  $N = N_P(V)$  and by the identity element  $i_N^N$  in  $\hat{\mathcal{F}}^{\mathfrak{y}}(N)$  belongs to  $\hat{\mathfrak{N}}(V)$ .

Then, for any pair of subgroups  $V$  and  $V'$  in  $\mathfrak{X} - \mathfrak{Y}$ , since for any  $(N, \hat{\nu}) \in \hat{\mathfrak{N}}(V)$  and any  $(N', \hat{\nu}') \in \hat{\mathfrak{N}}(V')$  the  $k^*$ -set  $\hat{\mathcal{F}}^{\mathfrak{x}}({}^{\nu'}V', {}^{\nu}V)$  is already defined, we denote by  $\hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  the  $k^*$ -subset of the direct product

$$\prod_{\hat{\nu} \in \hat{\mathfrak{N}}(V)} \prod_{\hat{\nu}' \in \hat{\mathfrak{N}}(V')} \hat{\mathcal{F}}^{\mathfrak{x}}({}^{\nu'}V', {}^{\nu}V) \quad 3.7.58$$

formed by the families  $\{\hat{\varphi}_{\hat{\nu}', \hat{\nu}}\}_{\hat{\nu} \in \hat{\mathfrak{N}}(V), \hat{\nu}' \in \hat{\mathfrak{N}}(V')}$  fulfilling

$$\hat{g}_{\hat{\nu}', \hat{\nu}} \cdot \hat{\varphi}_{\hat{\nu}', \hat{\nu}} = \hat{\varphi}_{\hat{\nu}', \hat{\nu}} \cdot \hat{g}_{\hat{\nu}', \hat{\nu}} \quad 3.7.59.$$

In other words, the set  $\hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  is the *inverse limit* of the family formed by the  $k^*$ -sets  $\hat{\mathcal{F}}^{\mathfrak{x}}(\nu' V', \nu V)$  and by the bijections between them induced by the  $\hat{\mathcal{F}}^{\mathfrak{x}}$ -isomorphisms  $\hat{g}_{\hat{\nu}, \hat{\nu}}$  and  $\hat{g}_{\hat{\nu}', \hat{\nu}'}$ .

Note that, according to equalities 3.7.57, the *projection map* onto the factor labeled by the pair  $((N, \hat{\nu}), (N', \hat{\nu}'))$  induces a  $k^*$ -set isomorphism

$$\hat{\mathfrak{n}}_{\hat{\nu}', \hat{\nu}} : \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \cong \hat{\mathcal{F}}^{\mathfrak{x}}(\nu' V', \nu V) \quad 3.7.60;$$

in particular, if  $V$  and  $V'$  are fully normalized in  $\mathcal{F}$ , setting  $N = N_P(V)$  and  $N' = N_P(V')$ , the pairs  $(N, \hat{i}_N^N)$  and  $(N', \hat{i}_{N'}^{N'})$  respectively belong to  $\hat{\mathfrak{N}}(V)$  and to  $\hat{\mathfrak{N}}(V')$ , and therefore we have a *canonical* bijection

$$\hat{\mathfrak{n}}_{\hat{i}_{N'}^{N'}, \hat{i}_N^N} : \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \cong \hat{\mathcal{F}}^{\mathfrak{x}}(\hat{i}_{N'}^{N'} V', \hat{i}_N^N V) \quad 3.7.61,$$

so that our notation is coherent. Moreover, we have an obvious map

$$\hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \longrightarrow \mathcal{F}(V', V) \quad 3.7.62.$$

Analogously, for any pair of subgroups  $Q$  and  $Q'$  of  $P$  respectively normalizing and strictly containing  $V$  and  $V'$ , we can define an injective  $k^*$ -set homomorphism

$$\hat{r}_{V', V}^{Q', Q} : \hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V} \longrightarrow \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \quad 3.7.63$$

which lifts the restriction map (cf. 3.2.1)

$$r_{V', V}^{Q', Q} : \mathcal{F}(Q', Q)_{V', V} \longrightarrow \mathcal{F}(V', V) \quad 3.7.64$$

and coincides with the  $k^*$ -set homomorphism 3.7.27 whenever  $V$  and  $V'$  are fully normalized in  $\mathcal{F}$ ; indeed, it is clear that we have suitable pairs  $(Q, \hat{\nu})$  in  $\hat{\mathfrak{N}}(V)$  and  $(Q', \hat{\nu}')$  in  $\hat{\mathfrak{N}}(V')$ , and then, for any  $\hat{\varphi} \in \hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V}$ , we set

$$\hat{\mathfrak{n}}_{\hat{\nu}', \hat{\nu}}(\hat{r}_{V', V}^{Q', Q}(\hat{\varphi})) = \hat{r}_{\hat{\nu}', \hat{\nu}}^{\nu' Q', \nu Q}(\hat{\nu}' \cdot \hat{\varphi} \cdot \hat{\nu}^{-1}) \quad 3.7.65,$$

which does not depend on our choices. Moreover, it is easily checked that equality 3.7.34 still holds in this general situation.

On the other hand, for any  $V'' \in \mathfrak{X} - \mathfrak{Y}$ , the  $k^*$ -composition map defined in 3.7.22 — and just noted  $\cdot$  from now on — can be extended to a new  $k^*$ -composition map

$$\hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') \times \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \longrightarrow \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V) \quad 3.7.66$$

sending  $(\hat{\varphi}', \hat{\varphi}) \in \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') \times \hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  to

$$\hat{\varphi}' \cdot \hat{\varphi} = (\hat{\mathfrak{n}}_{\hat{\nu}'', \hat{\nu}})^{-1}(\hat{\mathfrak{n}}_{\hat{\nu}'', \hat{\nu}'}(\hat{\varphi}') \cdot \hat{\mathfrak{n}}_{\hat{\nu}', \hat{\nu}}(\hat{\varphi})) \quad 3.7.67$$

for a choice of  $(N, \hat{\nu})$  in  $\hat{\mathfrak{N}}(V)$ , of  $(N', \hat{\nu}')$  in  $\hat{\mathfrak{N}}(V')$  and of  $(N'', \hat{\nu}'')$  in  $\hat{\mathfrak{N}}(V'')$ . This  $k^*$ -composition map does not depend on our choice; indeed, for another choice of pairs  $(\bar{N}, \hat{\nu}) \in \hat{\mathfrak{N}}(V)$ ,  $(\bar{N}', \hat{\nu}') \in \hat{\mathfrak{N}}(V')$  and  $(\bar{N}'', \hat{\nu}'') \in \hat{\mathfrak{N}}(V'')$ , we get (cf. 3.7.57)

$$\begin{aligned} \hat{g}_{\hat{\nu}'', \hat{\nu}'} \cdot (\hat{n}_{\hat{\nu}'', \hat{\nu}'}(\hat{\varphi}') \cdot \hat{n}_{\hat{\nu}', \hat{\nu}}(\hat{\varphi})) &= \hat{n}_{\hat{\nu}'', \hat{\nu}'}(\hat{\varphi}') \cdot \hat{g}_{\hat{\nu}', \hat{\nu}} \cdot \hat{n}_{\hat{\nu}', \hat{\nu}}(\hat{\varphi}) \\ &= \hat{n}_{\hat{\nu}'', \hat{\nu}'}(\hat{\varphi}') \cdot \hat{n}_{\hat{\nu}', \hat{\nu}}(\hat{\varphi}) \cdot \hat{g}_{\hat{\nu}', \hat{\nu}} = \hat{n}_{\hat{\nu}'', \hat{\nu}}(\hat{\varphi}' \cdot \hat{\varphi}) \cdot \hat{g}_{\hat{\nu}', \hat{\nu}} \end{aligned} \quad 3.7.68.$$

In particular, for any triple of subgroups  $Q$ ,  $Q'$  and  $Q''$  of  $P$  respectively normalizing and strictly containing  $V$ ,  $V'$  and  $V''$ , choosing pairs  $(Q, \hat{\nu})$  in  $\hat{\mathfrak{N}}(V)$ ,  $(Q', \hat{\nu}')$  in  $\hat{\mathfrak{N}}(V')$  and  $(Q'', \hat{\nu}'')$  in  $\hat{\mathfrak{N}}(V'')$ , the commutativity of the corresponding diagram 3.7.37 forces the commutativity of the analogous diagram in the general situation

$$\begin{array}{ccc} \hat{\mathcal{F}}^{\mathfrak{y}}(Q'', Q')_{V'', V'} \times \hat{\mathcal{F}}^{\mathfrak{y}}(Q', Q)_{V', V} & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{y}}(Q'', Q)_{V'', V} \\ \hat{r}_{V'', V'}^{Q'', Q'} \times \hat{r}_{V', V}^{Q', Q} \downarrow & & \downarrow \hat{r}_{V'', V}^{Q'', Q} \\ \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V') \times \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) & \longrightarrow & \hat{\mathcal{F}}^{\mathfrak{x}}(V'', V) \end{array} \quad 3.7.69.$$

Finally, for any  $V''' \in \mathfrak{X} - \mathfrak{Y}$  and any  $\hat{\varphi}'' \in \hat{\mathcal{F}}^{\mathfrak{x}}(V''', V'')$ , it follows from 3.7.25 that

$$(\hat{\varphi}'' \cdot \hat{\varphi}') \cdot \hat{\varphi} = \hat{\varphi}'' \cdot (\hat{\varphi}' \cdot \hat{\varphi}) \quad 3.7.70.$$

We are ready to complete our construction of the announced regular central  $k^*$ -extension  $\hat{\mathcal{F}}^{\mathfrak{x}}$  of  $\mathcal{F}^{\mathfrak{x}}$ ; we are already assuming that  $\hat{\mathcal{F}}^{\mathfrak{x}}$  contains  $\hat{\mathcal{F}}^{\mathfrak{y}}$  as a full  $k^*$ -subcategory over  $\mathfrak{Y}$ . For any subgroups  $V$  in  $\mathfrak{X} - \mathfrak{Y}$  and  $Q$  in  $\mathfrak{Y}$  we define

$$\hat{\mathcal{F}}^{\mathfrak{x}}(V, Q) = \emptyset \quad \text{and} \quad \hat{\mathcal{F}}^{\mathfrak{x}}(Q, V) = \bigsqcup_{V'} {}^Q \hat{\mathcal{F}}^{\mathfrak{x}}(V', V) \quad 3.7.71$$

where  $V'$  runs over the set of subgroups  $V' \in \mathfrak{X} - \mathfrak{Y}$  contained in  $Q$  and the  $k^*$ -subset  ${}^Q \hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  of  $\hat{\mathcal{F}}^{\mathfrak{x}}(Q, V)$  is a copy of the  $k^*$ -subset  $\hat{\mathcal{F}}^{\mathfrak{x}}(V', V)$  (cf. 3.7.16), coinciding with the converse image of the subset  $\iota_{V'}^Q \circ \mathcal{F}(V', V)$  of  $\mathcal{F}(Q, V)$ , so that we have a canonical map (cf. 3.7.17)

$$\hat{\mathcal{F}}^{\mathfrak{x}}(Q, V) \longrightarrow \mathcal{F}(Q, V) \quad 3.7.72;$$

in particular, if  $V' = V$  then we define  $\hat{\iota}_V^Q$  equal to the image  ${}^Q \hat{\iota}_V^V$  of  $\iota_V^V$  (cf. 3.7.17) in  ${}^Q \hat{\mathcal{F}}^{\mathfrak{x}}(V)$ .

In order to define the composition of two  $\hat{\mathcal{F}}^{\mathfrak{x}}$ -morphisms  $\hat{\varphi}: R \rightarrow Q$  and  $\hat{\psi}: T \rightarrow R$ , since we assume that  $\hat{\mathcal{F}}^{\mathfrak{x}}$  contains  $\hat{\mathcal{F}}^{\mathfrak{y}}$ , we already may assume that  $T$  does not belong to  $\mathfrak{Y}$ ; if  $Q$  does not belong to  $\mathfrak{Y}$  then the composition  $\hat{\varphi} \cdot \hat{\psi}$  is given by the map 3.7.67; if  $Q \in \mathfrak{Y}$  but  $R$  does not belong to  $\mathfrak{Y}$  then, setting  $R' = \varphi(R)$  where  $\varphi$  is the image of  $\hat{\varphi}$  in  $\mathcal{F}(Q, R)$ ,



it follows from definition 3.7.71 that  $\hat{\varphi}$  actually belongs to  ${}^Q\hat{\mathcal{F}}^x(R', R)$ , so that is the image of an element  $\hat{\varphi}^Q$  of  $\hat{\mathcal{F}}^x(R', R)$ ; since  $\hat{\psi}$  belongs to  $\hat{\mathcal{F}}^x(R, T)$ , the composition  $\hat{\varphi}^Q \cdot \hat{\psi}$  is defined by 3.7.67 and we can define the composition of  $\hat{\varphi}$  and  $\hat{\psi}$  as the image  ${}^Q(\hat{\varphi}^Q \cdot \hat{\psi})$  of  $\hat{\varphi}^Q \cdot \hat{\psi}$  in  ${}^Q\hat{\mathcal{F}}^x(Q, T)$ .

Finally, assume that  $R$  belongs to  $\mathfrak{Y}$  and, denoting by  $\psi$  the image of  $\hat{\psi}$  in  $\mathcal{F}(R, T)$ , consider the subgroups  $T' = \psi(T)$  of  $R$  and  $T'' = \varphi(T')$  of  $Q$ ; then, it follows again from the definition 3.7.71 that  $\hat{\psi}$  is actually the image of an element  $\hat{\psi}^R$  of  $\hat{\mathcal{F}}^x(T', T)$ ; moreover, setting  $\bar{R} = N_R(T')$  and  $\bar{Q} = N_Q(T'')$ , it is clear that  $\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi})$  belongs to  $\hat{\mathcal{F}}^{\mathfrak{Y}}(\bar{Q}, \bar{R})$  (cf. 3.6), so that  $\hat{r}_{\bar{T}'', \bar{T}'}^{\bar{Q}, \bar{R}}(\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi}))$  belongs to  $\hat{\mathcal{F}}^x(T'', T')$  (cf. 3.7.65); hence, we can define the composition  $\hat{\varphi} \cdot \hat{\psi}$  as the image of  $\hat{r}_{\bar{T}'', \bar{T}'}^{\bar{Q}, \bar{R}}(\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi})) \cdot \hat{\psi}^R$  (cf. 3.7.67) in  ${}^Q\hat{\mathcal{F}}^x(T'', T)$ .

This composition is clearly compatible with the action of  $k^*$ . Moreover, for a third  $\hat{\mathcal{F}}^x$ -morphism  $\hat{\theta}: V \rightarrow T$  we claim that

$$(\hat{\varphi} \cdot \hat{\psi}) \cdot \hat{\theta} = \hat{\varphi} \cdot (\hat{\psi} \cdot \hat{\theta}) \quad 3.7.73.$$

Once again, we may assume that  $V$  does not belong to  $\mathfrak{Y}$ ; if  $Q$  does not belong to  $\mathfrak{Y}$  then this equality follows from equality 3.7.70; if  $Q$  belongs to  $\mathfrak{Y}$  but  $R$  does not belong to  $\mathfrak{Y}$  then, with the notation above,  $\hat{\varphi}$  is actually the image of an element  $\hat{\varphi}^Q$  of  $\hat{\mathcal{F}}^x(R', R)$  and we have

$$\hat{\varphi} \cdot \hat{\psi} = {}^Q(\hat{\varphi}^Q \cdot \hat{\psi}) \quad 3.7.74;$$

moreover, with obvious notation we have  ${}^Q(\hat{\varphi}^Q \cdot \hat{\psi}) \cdot \hat{\theta} = {}^Q(\hat{\varphi}^Q \cdot (\hat{\psi} \cdot \hat{\theta}))$  and therefore we get (cf. 3.7.70)

$$(\hat{\varphi} \cdot \hat{\psi}) \cdot \hat{\theta} = {}^Q(\hat{\varphi}^Q \cdot (\hat{\psi} \cdot \hat{\theta})) = \hat{\varphi} \cdot (\hat{\psi} \cdot \hat{\theta}) \quad 3.7.75.$$

From now on, we assume that  $R$  belongs to  $\mathfrak{Y}$ , denote by  $\theta$  the image of  $\hat{\theta}$  in  $\mathcal{F}(T, V)$ , consider the subgroups  $V' = \theta(V)$  of  $T$ ,  $V'' = \psi(V')$  of  $R$  and  $V''' = \varphi(V'')$  of  $Q$ , and set  $\bar{T} = N_T(V')$ ,  $\bar{R} = N_R(V'')$  and  $\bar{Q} = N_Q(V''')$ ; if  $T$  belongs to  $\mathfrak{Y}$  then, with the notation above, we get

$$\begin{aligned} (\hat{\varphi} \cdot \hat{\psi}) \cdot \hat{\theta} &= {}^Q\left(\hat{r}_{\bar{V}''', \bar{V}'}^{\bar{Q}, \bar{T}}(\hat{r}_{\bar{Q}, \bar{T}}^{Q, T}(\hat{\varphi} \cdot \hat{\psi})) \cdot \hat{\theta}^T\right) \\ &= {}^Q\left(\hat{r}_{\bar{V}''', \bar{V}'}^{\bar{Q}, \bar{T}}(\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi}) \cdot \hat{r}_{\bar{R}, \bar{T}}^{R, T}(\hat{\psi})) \cdot \hat{\theta}^T\right) \\ &= {}^Q\left(\left(\hat{r}_{\bar{V}''', \bar{V}''}^{\bar{Q}, \bar{R}}(\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi})) \cdot \hat{r}_{\bar{V}'', \bar{V}'}^{\bar{R}, \bar{T}}(\hat{r}_{\bar{R}, \bar{T}}^{R, T}(\hat{\psi}))\right) \cdot \hat{\theta}^T\right) \\ &= {}^Q\left(\hat{r}_{\bar{V}''', \bar{V}''}^{\bar{Q}, \bar{R}}(\hat{r}_{\bar{Q}, \bar{R}}^{Q, R}(\hat{\varphi})) \cdot \left(\hat{r}_{\bar{V}'', \bar{V}'}^{\bar{R}, \bar{T}}(\hat{r}_{\bar{R}, \bar{T}}^{R, T}(\hat{\psi})) \cdot \hat{\theta}^T\right)\right) \end{aligned} \quad 3.7.76;$$

but, according to our definition above, we have

$$\hat{r}_{V'',V'}^{\bar{R},\bar{T}}(\hat{r}_{\bar{R},\bar{T}}^{R,T}(\hat{\psi})) \cdot \hat{\theta}^T = (\hat{\psi} \cdot \hat{\theta})^R \quad 3.7.77$$

and similarly, we also get

$$(\hat{\varphi} \cdot \hat{\psi}) \cdot \hat{\theta} = {}^Q \left( \hat{r}_{V''',V''}^{\bar{Q},\bar{R}}(\hat{r}_{\bar{Q},\bar{R}}^{Q,R}(\hat{\varphi})) \cdot (\hat{\psi} \cdot \hat{\theta})^R \right) = \hat{\varphi} \cdot (\hat{\psi} \cdot \hat{\theta}) \quad 3.7.78,$$

which proves equality 3.7.73 and completes the proof of the existence of  $\hat{\mathcal{F}}^x$ .

If  $\hat{\mathcal{F}}^x$  is another central regular  $k^*$ -extension of  $\mathcal{F}^x$  inducing the same *folder structure* and we have  $N_{\mathcal{F}}(U) \neq \mathcal{F}$ , denoting by  $\hat{\mathcal{F}}^y$  the *full*  $k^*$ -subcategory of  $\hat{\mathcal{F}}^x$  over  $\mathfrak{Y}$  and applying our induction hypothesis and Lemma 3.5, we may assume that  $\hat{\mathcal{F}}^y = \hat{\mathcal{F}}^y$ . Once again, for any  $V \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$  we consider the same *regular central  $k^*$ -extension*  $\widehat{N_{\mathcal{F}}(V)}^{\text{nc}}$  of  $N_{\mathcal{F}}(V)^{\text{nc}}$  determining  $\widehat{\text{aut}}_{N_{\mathcal{F}}(V)^{\text{nc}}}$  and the two  $\mathfrak{ch}^*(N_{\mathcal{F}}(V)^{\text{nc}})$ -morphisms in 3.7.14; since  $\hat{\mathcal{F}}^x$  and  $\hat{\mathcal{F}}^x$  induce the same *folder structure*, we get the same  $k^*$ -group homomorphisms 3.7.15 and  $\hat{\mathcal{F}}^x(V)$  necessarily coincides with  $\widehat{\mathcal{F}}(V)$ . Hence, since isomorphism 3.3.1 forces a canonical  $k^*$ -group isomorphism

$$\hat{\mathcal{F}}^x(V', V) \cong \hat{\mathcal{F}}^y(N', N)_{V',V} \times_{\hat{\mathcal{F}}^y(N)_V} \hat{\mathcal{F}}^x(V) \quad 3.7.79$$

induced by the inclusion of  $\hat{\mathcal{F}}^y(N', N)_{V',V}$  in and the action of  $\hat{\mathcal{F}}^x(V)$  on  $\hat{\mathcal{F}}^x(V', V)$ , definition 3.7.16 determines a  $k^*$ -set bijection

$$\hat{\mathcal{F}}^x(V', V) \cong \hat{\mathcal{F}}^x(V', V) \quad 3.7.80$$

compatible with both canonical maps to  $\mathcal{F}(V', V)$  (cf. 3.7.17).

For another  $V'' \in \mathfrak{X} - \mathfrak{Y}$  fully normalized in  $\mathcal{F}$  we also have  $k^*$ -set bijections

$$\hat{\mathcal{F}}^x(V'', V) \cong \hat{\mathcal{F}}^x(V'', V) \quad \text{and} \quad \hat{\mathcal{F}}^x(V'', V') \cong \hat{\mathcal{F}}^x(V'', V') \quad 3.7.81$$

and it is clear that the composition in  $\hat{\mathcal{F}}^x$  induces a  $k^*$ -set homomorphism

$$\hat{c}_{V'',V',V}^x : \hat{\mathcal{F}}^x(V'', V') \times \hat{\mathcal{F}}^x(V', V) \longrightarrow \hat{\mathcal{F}}^x(V'', V) \quad 3.7.82;$$

then, since we get the *same* commutative diagrams of  $k^*$ -groups 3.7.21, it is easily checked from definition 3.7.22 that we obtain an obvious commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{F}}^x(V'', V') \times \hat{\mathcal{F}}^x(V', V) & \longrightarrow & \hat{\mathcal{F}}^x(V'', V) \\ \parallel & \times & \parallel \\ \hat{\mathcal{F}}^x(V'', V') \times \hat{\mathcal{F}}^x(V', V) & \longrightarrow & \hat{\mathcal{F}}^x(V'', V) \end{array} \quad 3.7.83.$$

Moreover, if  $Q$  and  $Q'$  are a pair of subgroups of  $P$  respectively contained in  $N = N_P(V)$  and  $N' = N_P(Q')$ , and strictly containing  $V$  and  $V'$ , recall that in  $\widehat{\mathcal{F}}^{\mathfrak{X}}$  we have a  $k^*$ -restriction map (cf. 3.6)

$$\widehat{r}_{V',V}^{Q',Q} : \widehat{\mathcal{F}}^{\mathfrak{Y}}(Q', Q)_{V',V} \longrightarrow \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) \quad 3.7.84$$

lifting the restriction map (cf. 3.2.1); then, it is easily checked from definition 3.7.32 that we still have the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{F}}^{\mathfrak{Y}}(Q', Q)_{V',V} & \longrightarrow & \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) \\ \parallel & & \parallel \\ \widehat{\mathcal{F}}^{\mathfrak{Y}}(Q', Q)_{V',V} & \longrightarrow & \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) \end{array} \quad 3.7.85.$$

Finally, for any subgroup  $V$  in  $\mathfrak{X} - \mathfrak{Y}$  note that the set of pairs  $\widehat{\mathfrak{N}}(V)$  defined in 3.7.53 above coincides with the analogous set of pairs  $\widehat{\mathfrak{N}}(V)$  corresponding to  $\widehat{\mathcal{F}}^{\mathfrak{X}}$  since both only depend on  $\widehat{\mathcal{F}}^{\mathfrak{Y}}$ ; moreover, for any pair of elements  $(N, \hat{\nu})$  and  $(\bar{N}, \hat{\nu})$  in  $\widehat{\mathfrak{N}}(V)$ , it is easily checked that the corresponding element  $\widehat{g}_{\hat{\nu}, \hat{\nu}}$  defined in 3.7.54 coincides with the restriction  $\widehat{r}_{\hat{\nu}V, \hat{\nu}V}^{\hat{\nu}\bar{N}, \hat{\nu}N}(\hat{\nu} \cdot \hat{\nu}^{-1})$  which is indeed already defined in  $\widehat{\mathcal{F}}^{\mathfrak{X}}$ . Similarly, for another subgroup  $V'$  in  $\mathfrak{X} - \mathfrak{Y}$  and an element  $(N', \hat{\nu}')$  in  $\widehat{\mathfrak{N}}(V')$ , we have an obvious  $k^*$ -set isomorphism

$$\widehat{n}_{\hat{\nu}', \hat{\nu}} : \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) \cong \widehat{\mathcal{F}}^{\mathfrak{X}}(\hat{\nu}' V', \hat{\nu} V) \quad 3.7.86$$

induced by the corresponding restrictions of  $\hat{\nu}$  and  $\hat{\nu}'$ ; then, it is easily checked from definition 3.7.60 and bijections 3.7.80 that we also get the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) & \cong & \widehat{\mathcal{F}}^{\mathfrak{X}}(\hat{\nu}' V', \hat{\nu} V) \\ \parallel & & \parallel \\ \widehat{\mathcal{F}}^{\mathfrak{X}}(V', V) & \cong & \widehat{\mathcal{F}}^{\mathfrak{X}}(\hat{\nu}' V', \hat{\nu} V) \end{array} \quad 3.7.87.$$

Now, it is quite clear that the commutative diagram 3.7.85 above remains true for any pair of subgroups  $V$  and  $V'$  in  $\mathfrak{X} - \mathfrak{Y}$ .

At this point, from our definition 3.7.71 and bijections 3.7.80, for any subgroups  $V$  in  $\mathfrak{X} - \mathfrak{Y}$  and  $Q$  in  $\mathfrak{Y}$  we obtain evident  $k^*$ -set bijections

$$\widehat{\mathcal{F}}^{\mathfrak{X}}(V, Q) \cong \widehat{\mathcal{F}}^{\mathfrak{X}}(V, Q) \quad \text{and} \quad \widehat{\mathcal{F}}^{\mathfrak{X}}(Q, V) \cong \widehat{\mathcal{F}}^{\mathfrak{X}}(Q, V) \quad 3.7.88;$$

that is to say, for any pair of subgroups  $Q$  and  $R$  in  $\mathfrak{X}$  we have obtained a  $k^*$ -set bijection

$$\widehat{\mathcal{F}}^{\mathfrak{X}}(Q, R) \cong \widehat{\mathcal{F}}^{\mathfrak{X}}(Q, R) \quad 3.7.89$$

and it is routine to check that all these bijections are compatible with both compositions. We are done.

3.8. As in 1.1 above, consider the Frobenius  $P$ -category  $\mathcal{F}_{(b,G)}$  associated with a block  $b$  of a finite group  $G$ , having  $P$  as a defect  $p$ -subgroup; recall that from [4, Theorem 11.32] we get a so-called *Brauer folder structure* of  $\mathcal{F}_{(b,G)}$ ; thus, from the theorem above, we obtain the following result.

**Corollary 3.9.** *With the notation above, the Brauer folder structure of  $\mathcal{F}_{(b,G)}$  determines a regular central  $k^*$ -extension  $\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}$  of  $\mathcal{F}_{(b,G)}^{\text{nc}}$  admitting a  $k^*$ -group isomorphism*

$$\hat{\mathcal{F}}_{(b,G)}^{\text{nc}}(Q) \cong \hat{N}_G(Q, f)/C_G(Q) \quad 3.9.1$$

for any  $\mathcal{F}_{(b,G)}$ -nilcentralized subgroup  $Q$  of  $P$ .

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